Jacob Lurie's 114. TF: Stephen Mackereth. Solution set 1. Aubrey Clark

Problem 1:

For each natural number $k > 0$ consider $\frac{2}{\pi}\int_0^\pi f(x)^2 dx - \sum_{n=1}^k a_n^2$. This can be rewritten as

$$\frac{2}{\pi}\int_0^\pi f(x)^2 dx - 2\sum_{n=1}^k a_n^2 + \sum_{n=1}^k a_n^2,$$

which can be rewritten as

$$\frac{2}{\pi}\int_0^\pi \left(f(x) - \sum_{n=1}^k a_n \sin(nx)\right)^2 dx,$$

and this is at least zero. This shows that $\frac{2}{\pi}\int_0^\pi f(x)^2 dx \geq \sum_{n=1}^k a_n^2$. So $\sum_{n=1}^k a_n^2$ converges.

Problem 2:

Using the sequences

$$\cos\left(\frac{n\pi}{4}\right) : \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}, -1, -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 1, \text{ repeating,}$$

$$\cos\left(\frac{3n\pi}{4}\right) : -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, -1, \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}, 1, \text{ repeating,}$$

$$\cos\left(\frac{n\pi}{4}\right) - \cos\left(\frac{3n\pi}{4}\right) : \sqrt{2}, 0, -\sqrt{2}, 0, -\sqrt{2}, 0, \sqrt{2}, 0, \text{ repeating,}$$

we have that a_n which equals $\frac{2}{\pi}\int_0^\pi \sin(nx) f(x) dx$ is given by

$$\frac{2}{\pi}\int_0^\pi \sin(nx) f(x) dx : \frac{2\sqrt{2}}{\pi}, 0, -\frac{2\sqrt{2}}{3\pi}, 0, -\frac{2\sqrt{2}}{5\pi}, 0, \frac{2\sqrt{2}}{7\pi}, 0, \text{ repeating.}$$

Using this and the sequence

$$a_n \sin\left(\frac{n\pi}{4}\right) : \frac{2}{\pi}, 0, -\frac{2}{3\pi}, 0, \frac{2}{5\pi}, 0, -\frac{2}{7\pi}, 0, \text{ repeating,}$$

we have

$$\sum_{n>0} a_n \sin\left(\frac{n\pi}{4}\right) = \frac{2}{\pi}\left(\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots\right) = \frac{2}{\pi}\tan^{-1}(1) = \frac{1}{2}.$$

Likewise, using the sequence

$$a_n \sin\left(\frac{3n\pi}{4}\right) : \frac{2}{\pi}, 0, -\frac{2}{3\pi}, 0, \frac{2}{5\pi}, 0, -\frac{2}{7\pi}, 0, \text{repeating,}$$

we have

$$\sum_{n>0} a_n \sin\left(\frac{3n\pi}{4}\right) = \frac{2}{\pi}\left(\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots\right) = \frac{2}{\pi}\tan^{-1}(1) = \frac{1}{2}.$$

Problem 3:

Let V be a k-dimensional subspace of $\mathbf{R}^n$ where $k < n$. Consider the k-dimensional square $[0,1] \times [0,1] \times \cdots \times [0,1]$. It is possible to contain this k-dimensional square in the union of $(m+1)^k$ n-dimensional open boxes with edges of length $\frac{1}{m}$. The measure of each of these n-dimensional open boxes is $\frac{1}{m^n}$. This implies that the outer measure of this k-dimensional square is no more than $\frac{(m+1)^k}{m^n}$. This can be made as close to zero as we like by making m big. It follows that the outer measure of the k-dimensional square is zero. We may rotate and translate this k-dimensional square so that it is a subset of V; we may also translate our open boxes so that they cover it. Since the volume of an open box is the same as the volume of the open box translated the outer measure of a translated rotation of the k-dimensional square is zero as well. Now tile $\mathbf{R}^k$ with k-dimensional squares. Each square in the tiling contains a point with rational coordinates that the others do not. So there are countably many. Consider a rotation of the union of this tiling that makes it a subset of V. By subadditivity the outer measure of V is zero.