Jacob Lurie's 114. TF: Stephen Mackereth. Solution set 2. Aubrey Clark

Problem 1: Let $S \subseteq \mathbf{R}^n$ be a measurable set with $\mu(S) < \infty$, and let $\epsilon > 0$ be a positive real number. Show that there exists a compact subset $K \subseteq S$ such that

$$\mu(S) - \epsilon \leq \mu(K) \leq \mu(S).$$

Solution 1: Since S is bounded there exists a closed set C that contains S in its interior. Let $B_1, B_2, \cdots$ be open boxes inside C whose union (which I'll denote by A) contains $B \cap S^c$ such that

$$\mu^*(A) \leq \mu^*(C \cap S^c) + \epsilon.$$

Since A and S are measurable this can be written as

$$\mu^*(C) - \mu^*(C \cap A^c) \leq \mu^*(C) - \mu^*(S) + \epsilon$$

so that

$$\mu^*(C \cap A^c) \geq \mu^*(S) - \epsilon.$$

The set $C \cap A^c$ is a compact subset of S. Finally, $\mu^*(S) \geq \mu^*(C \cap A^c)$ by the monotonicity of the outer measure.

Problem 2: Let X be a set and let $\mathcal{M}$ be a σ-algebra of subsets of X. Suppose that $m : \mathcal{M} \to [0, \infty]$ is a function satisfying the following axioms:

1. The function m is finely additive. That is, if $S, T \subseteq X$ are disjoint sets belonging to $\mathcal{M}$, then $m(S \cup T) = m(S) + m(T)$.
2. The function m is countably subadditive. That is, for every sequence of subsets $S_0, S_1, \cdots \subseteq X$ which belong to $\mathcal{M}$, we have an inequality

$$m\left(\bigcup_{n \geq 0} S_n\right) \leq \sum_{n \geq 0} m(S_n).$$

Show that m is countably additive. That is, if $S_0, S_1, \cdots \subseteq X$ is a sequence of pairwise disjoint subsets of X which belong to $\mathcal{M}$, show that $m\left(\bigcup_{n \geq 0} S_n\right) = \sum_{n \geq 0} m(S_n)$.

Solution 2: I will first show that m is monotone. That is, if S and T are subsets of X such that $S \subseteq T$, then $m(S) \leq m(T)$. This follows from the finite additivity of m:

$$m(T) = m(S) + m(T - S) \geq m(S).$$

Now I'll show that $m\left(\bigcup_{n \geq 0} S_n\right) \geq \sum_{n \geq 0} m(S_n)$. For each natural number k we have by monotonicity and finite subadditivity of m that

$$m\left(\bigcup_{n\geq 0} S_n\right) \geq m\left(\bigcup_{n=0}^{k} S_n\right) = \sum_{n\geq 0}^{k} m(S_n).$$

The result follows because this inequality holds for all k.

Finally, the reverse inequality, $m\left(\bigcup_{n\geq 0} S_n\right) \leq \sum_{n\geq 0} m(S_n)$, it true by the subadditivity of m.

Problem 3: Let E be a subset of $\mathbf{R}^n$. Show that E is measurable if and only if

$$\mu^*(B\cap E) + \mu^*(B\cap E^c) = \mu^*(B),$$

for every open box $B \subseteq \mathbf{R}^n$.

Solution 3: That

$$\mu^*(B\cap E) + \mu^*(B\cap E^c) = \mu^*(B),$$

for every open box $B \subseteq \mathbf{R}^n$ whenever E is measurable is clear from the definition of measurability.

For the other implication, let S be a subset of $\mathbf{R}^n$. Then $\mu^*(S\cap E) + \mu^*(S\cap E^c) \geq \mu^*(S)$ by subadditivity of the outer measure. For the reverse inequality we have that for each positive natural number n there exist open boxes $B_1, B_2, \cdots$ whose union contains S such that

$$\begin{aligned}\mu^*(S) + \frac{1}{n} \geq \sum_{k=1}^{\infty} \mu^*(B_k) = \sum_{k=1}^{\infty} \mu^*(B_k\cap E) + \sum_{k=1}^{\infty} \mu^*(B_k\cap E^c)\\ \geq \mu^*\left((B_1\cup B_2\cup\cdots)\cap E\right) + \mu^*((B_1\cup B_2\cup\cdots)\cap E^c)\\ \geq \mu^*(S\cap E) + \mu^*(S\cap E^c).\end{aligned}$$

Since this holds for all n, we have that $\mu^*(S) \geq \mu^*(S\cap E) + \mu^*(S\cap E^c)$.

Problem 4: Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function. Recall that f is differentiable at a point $x \in \mathbf{R}$ if the expression $\frac{f(x+h)-f(x)}{h}$ approaches a limit as $h \to 0$. Show that $S = \{x \in \mathbf{R} \mid f \text{ is differentiable at } x\}$ is a Borel set.

Solution 4: We have that

$$S = \bigcap_{n\geq 1} \bigcup_{k\in\mathbf{Z}-\{0\}} \{x \in \mathbf{R} \mid \text{ there exists } y \text{ in } \mathbf{R} \text{ such that } \frac{f(x+\frac{1}{k})-f(x)}{\frac{1}{k}} \in (y-\frac{1}{n}, y+\frac{1}{n})\}.$$

The set in brackets is an open set because the formula $\frac{f(x+\frac{1}{k})-f(x)}{\frac{1}{k}}$ defines a continuous function in the variable x. Therefore S is a Borel set.