Jacob Lurie's 114. TF: Stephen Mackereth. Solution set 3. Aubrey Clark

Problem 1: Let $E \subseteq \mathbf{R}^n$ be a measurable set with $\mu(E) < \infty$. Show that for each $\epsilon > 0$, there exists a set $E' \subseteq \mathbf{R}^n$ which is a finite disjoint union of open boxes satisfying

$$\mu(E - E'), \mu(E' - E) < \epsilon.$$

Solution 1: Let δ_1 and δ_2 be positive real numbers. By problem 1 on last week's problem set there exists a compact set $K \subseteq E$ such that $\mu(E - K) < \delta_1$. Let $B_1, B_2, \cdots$ be open boxes whose union cover K such that $\mu(K) + \delta_2 \geq \mu(B_1) + \mu(B_2) + \cdots$. Since K is compact it is contained in the union of finitely many of the open boxes (say, $B_1, B_2, \cdots, B_n$). Then

$$\mu(K) + \delta_2 \geq \mu(B_1) + \mu(B_2) + \cdots \mu(B_n) \geq \mu(B_1 \cup B_2 \cup \cdots \cup B_n).$$

Denote the open box B_i by $(a_1^i, b_1^i) \times (a_2^i, b_2^i) \times \cdots (a_n^i, b_n^i)$. Let c_j^k be the kth biggest element of $\{a_j^1, b_j^1, a_j^2, b_j^2, \cdots, a_j^n, b_j^n\}$. Consider the collection of open boxes

$$\{C \mid C \text{ can be written as } (c_1^m, c_1^{m+1}) \times (c_2^t, c_2^{t+1}) \times \cdots \times (c_n^s, c_n^{s+1})\}$$

which I'll denote by $\mathcal{C}$.

This collection of open boxes is finite. Its union is a subset of $B_1 \cup B_2 \cup \cdots \cup B_n$ and the difference $(B_1 \cup B_2 \cup \cdots \cup B_2) - \bigcup \mathcal{C}$ has Lebesgue measure zero since it it the union of subsets of $\mathbf{R}^n$ contained in subspaces of dimension less than n. Thereore the Lebesgue measure of $B_1 \cup B_2 \cup \cdots B_n$ is equal to the Lebesgue measure of $\bigcup \mathcal{C}$. Let E' denote $\bigcup \mathcal{C}$. By countable addiivity and monotonicity of μ we have

$$\mu(E - E') = \mu(E - K) - \mu(E' - K) = \mu(E - K) - (\mu(E') - \mu(K \cap E'))$$

$$\leq \mu(E - K) + \mu(B_1 \cup B_2 \cup \cdots \cup B_n) - \mu(K) < \delta_1 + \delta_2.$$

Since we can choose $\delta_1 + \delta_2$ to be smaller than ϵ this proves the first inequality. Likewise

$$\mu(E' - E) = \mu(E' - K) - \mu(E - K) = \mu(E' - K) - \mu(E - K) = \mu(E') - (\mu(E' \cap K) - \mu(E - K))$$

$$\leq \mu(B_1 \cup B_2 \cup \cdots \cup B_n) - \mu(K) + \mu(E - K) < \delta_1 + \delta_2.$$

Problem 2: Let $f_1, f_2, \cdots : \mathbf{R}^n \to \mathbf{R}$ be a sequence of measurable functions and suppose that for each $x \in \mathbf{R}^n$, the sequence $\{f_i(x)\}$ is bounded. Show that the function $f(x) = \limsup\{f_i(x)\}$ is measurable.

Solution 2: The set $\{x \in \mathbf{R} \mid f(x) \leq t\}$ is measurable because it can be written as

$$\bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} \{x \in \mathbf{R} \mid f_i(x) \leq t + \frac{1}{k}\},$$

the countable intersection of the countable union of measurable sets.

Problem 3: Let $f : \mathbf{R} \to \mathbf{R}$ be a function. We say that f is **Borel measurable** if, for every real number t, the set $\{x \in \mathbf{R} \mid f(x) \leq t\}$ is Borel measurable. Prove that if $f, g : \mathbf{R} \to \mathbf{R}$ are Borel measurable functions, then the composition $g \circ f$ is Borel measurable.

Solution 3: The set $\{x \in \mathbf{R} \mid g \circ f(x) \leq t\}$ is the inverse image under f of the set $\{x \in \mathbf{R} \mid g(x) \leq t\}$. Since g is Borel measurable the set $\{x \in \mathbf{R} \mid g(x) \leq t\}$, which I'll denote by A, is a Borel set and so can be written as an expression consisting of complements and countable unions of open sets. Since the collection of intervals of the form (q, r) where q and r are rational numbers is a countable basis for the usual open sets of $\mathbf{R}$ we can write A as an expression consisting of complements and countable unions of such intervals. In this expression we can write each interval (q, r) as $(\infty, q]^c \cap \bigcup_{k=1}^{\infty}(-\infty, r - \frac{1}{k}]$. The inverse image of the set A by f is then an expression consisting of complements and countable unions of sets of the form $\{x \in \mathbf{R} \mid f(x) \leq r + \frac{1}{k}\}$ and $\{x \in \mathbf{R} \mid f(x) \leq q\}$. It follows that the set $\{x \in \mathbf{R} \mid g \circ f(x) \leq t\}$ is Borel measurable and so the function $g \circ f$ is Borel measurable.

Problem 4: Let $f : \mathbf{R} \to \mathbf{R}$ be a measurable function. Show that there exists a Borel measurable function g which is equal to f almost everywhere.

Solution 4: By Lusin's theorem there exists a continuous function $g_i : \mathbf{R} \to \mathbf{R}$ and a set E_i such that $\mu(\mathbf{R} - E_i) < \frac{1}{i}$ and $f(x) = g_i(x)$ whenever x belongs to E_i. Let $g(x)$ be the limit of the sequence of numbers $g_1(x), g_2(x), \cdots$. Then $g(x) = f(x)$ whenever x belongs to $\bigcup_{i=1}^{\infty} E_i$. Also, the same proof as in problem 2 shows that g is Borel measurable as the limit of continuous (and so Borel measurable) functions. Finally, the set $\mathbf{R} - \bigcup_{i=1}^{\infty} E_i$ (the only points where f may not equal g) has Lebesgue measure zero since for all integers i its measure is no more than $\frac{1}{i}$.