Jacob Lurie's 114. TFs: Stephen Mackereth, Patrick Ryan. Solution set 4. Aubrey Clark

Problem 1: Let $E \subseteq \mathbf{R}^n$ be a measurable set and let $f : E \to \mathbf{R}$ be a nonnegative measurable function. Show that the set $\{x \in E \mid f(x) \neq 0\}$ has measure zero if and only if $\int_E f = 0$.

Solution 1: The symbol $\int_E f$ denotes the supremum of the set of integrals of nonnegative simple functions $g : E \to \mathbf{R}$ that are less than or equal to f. That is,

$$\int_E f = \sup \left\{ \int g \;\middle|\; g : E \to \mathbf{R} \text{ is a simple function with } 0 \leq g \leq f \right\}.$$

If $\int_E f$ is positive, then there exists a simple function $g : E \to \mathbf{R}$ whose integral is positive. This implies that g is positive on a set of positive measure which implies that f is positive on a set of positive measure.

Conversely, suppose that the set $\{x \in E \mid f(x) \neq 0\}$ has positive measure. I claim that there then exists a positive natural number n such that the set $\{x \in E \mid f(x) \geq 1/n\}$ has positive measure. This is true by the continuity of measure property. Now define the simple function that takes a value of $1/n$ on the set $\{x \in E \mid f(x) \geq 1/n\}$ and zero elsewhere in E. The integral of this simple function is positive so $\int_E f$ is too.

Problem 2: Let $E \subseteq \mathbf{R}^n$ be a measurable set, and let $f_1, f_2, \cdots$ be a sequence of measurable functions on E which converge pointwise to another function $f : E \to \mathbf{R}^n$. Show that there exists a sequence of subsets $E_0 \subseteq E_1 \subseteq \cdots \subseteq E$ such that $\mu\left(E - \bigcup E_i\right) = 0$ and the sequence $\{f\,|_{E_j}\}$ converges uniformly to f_{E_j} for each j.

Solution 2: This is similar to Lusin's theorem, except here the set E may have infinite measure. Let B_j denote the open ball of radius j. Let $F_1 = E \cap B_1$, $F_2 = E \cap B_2$, and so on.

Consider the set F_j and the sequence of functions $\{f\,|_{F_j}\}$. Each function in this sequence is measurable and converges pointwise to the function $f\,|_{F_j}$. Let ϵ be a positive real number. By Lusin's theorem there exists a subset E_j of F_j such that $\mu(F_j - E_j) < \frac{\epsilon}{2^j}$ and such that the sequence $\{f\,|_{E_j}\}$ converges uniformly to $f\,|_{E_j}$. Doing this for each j we get that $\mu\left(E - \bigcup E_i\right) = 0$ because our choice of ϵ is any positive real number and

$$\mu\left(E - \bigcup E_i\right) = \mu\left(\bigcup F_i - \bigcup E_i\right) \leq \mu\left(\bigcup (F_i - E_i)\right) \leq \epsilon.$$

Problem 3: Let $f : \mathbf{R}^n \to \mathbf{R}$ be a measurable function. Show that for each $\epsilon > 0$, there exists a continuous function $g : \mathbf{R}^n \to \mathbf{R}$ such that the set $\{x \in \mathbf{R}^n \mid f(x) \neq g(x)\}$ has measure at most ϵ.

Solution 3: This is similar to Lusin's theorem, except now the function f is defined on a set of infinite measure. Lusin's theorem states that if f is a real valued measurable function on a subset E of $\mathbf{R}^n$ of finite measure and ϵ is a positive number, then there exists a continuous real valued function g on $\mathbf{R}^n$ and a closed set $E' \subseteq E$ such that $f = g$ on E' and $\mu(E - E') < \epsilon$

Let $B(k)$ denote the open ball in $\mathbf{R}^n$ centered at the origin and of radius k. Let $S(k) =$

$S(k)-S(k-1)$ for $k=1,2,\cdots$. Let f_k denote the function f restricted to the set $S(k)$. By our version of Lusin's theorem there exists a closed subset E_k of $S(k)$ and a continuous real valued function g_k on $\mathbf{R}^n$ such that $g_k = f_k$ on E_k and $\mu(S(k)-E_k) < \frac{\epsilon}{2^k}$.

Consider a pair of sets E_k and E_{k+1}. These sets are compact and disjoint and so the distance between their boundaries is positive. Define the real valued function $h_k : \mathbf{R}^n$ to take the value $tg_k(x_k) + (1-t)g_{k+1}(x_{k+1})$ at any point that can be expressed as $tx_k + (1-t)x_{k+1}$ where x_k is a boundary point of E_k, x_{k+1} is a boundary point of E_{k+1}, and t is a number in the unit interval.

Define the function $g : \mathbf{R}^n \to \mathbf{R}$ by saying that $f(x) = g_k(x)$ whenever x belongs to $S(k)$ and $f(x) = h_k(x)$ whenever x can be expressed as $tx_k + (1-t)x_{k+1}$ where x_k is a boundary point of E_k, x_{k+1} is a boundary point of E_{k+1}, and t is a number in the unit interval.

This function is clearly continuous. The set of points at which f differs from g, that is , the set $\{x \in \mathbf{R}^n \mid f(x) \neq g(x)\}$, must belong to the disjoint union of sets $\bigcup_{k=1}^{\infty}(B(k)-E_k)$. So by the monotonicity and countable additivity of Lebesgue measure,

$$\mu(\{x \in \mathbf{R}^n \mid f(x) \neq g(x)\}) \leq \mu\left(\bigcup_{k=1}^{\infty}(B(k)-E_k)\right) = \sum_{k=1}^{\infty}\frac{\epsilon}{2^k} = \epsilon.$$

Problem 4: Let $E \subseteq \mathbf{R}^m$ and $E' \subseteq \mathbf{R}^n$ be measurable sets. Show that $E \times E'$ is a measurable subset of $\mathbf{R}^{m+n}$, and that $\mu_{m+n}(E \times E') = \mu_m(E)\mu_n(E')$.

Solution 4: By problem 1 on last week's problem set there exists a set $S \subset \mathbf{R}^m$ which is a finite union of disjoint open boxes satisfying $\mu_m(E-S) < \epsilon, \mu_m(S-E) < \epsilon$. There also exists a set $S' \subset \mathbf{R}^n$ which is a finite union of disjoint open boxes satisfying $\mu_n(E'-S') < \epsilon, \mu_n(S'-E') < \epsilon$. It follows that set $S \times S'$ is a finite union of disjoint open boxes. It is therefore a measurable set. It is clear that $\mu_{m+n}(S \times S') = \mu_m(S)\mu_n(S')$. Let $S \times S'$ be a measurable subset of $\mathbf{R}^{n+m}$ such that $\mu_m(E \times E' - S \times S') < \epsilon, \mu_m(E \times E' - S \times S') < \epsilon$. Then

$$\mid \mu_{m+n}(E \times E') - \mu_m(E)\mu_n(E') \mid$$

$$\leq \mid \mu_{m+n}(E \times E') - \mu_{m+n}(S \times S') \mid + \mid \mu_m(E)\mu_n(E') - \mu_m(S)\mu_n(S') \mid$$

$$\leq \mid \mu_{m+n}(E \times E') - \mu_{m+n}(S \times S') \mid + \mid \mu_m(E) \mid \ \mid \mu_n(E') - \mu_n(S') \mid + \mid \mu_n(S') \mid \ \mid \mu_m(E) - \mu_m(S) \mid$$

If E and E' are sets of finite measure then $\mu_{m+n}(E \times E')$ can be made arbitrarily close to $\mu_m(E)\mu_n(E')$ by taking ϵ to be small. In this case $\mu_{m+n}(E \times E') = \mu_m(E)\mu_n(E')$. If one of E or E' has zero measure then both $\mu_{m+n}(E \times E') = \mu_m(E)\mu_n(E') = 0$; If they both have positive measure and one has infinite measure then $\mu_{m+n}(E \times E') = \mu_m(E)\mu_n(E') = \infty$.