Jacob Lurie's 114. TFs: Stephen Mackereth, Patrick Ryan. Solution set 10. Aubrey Clark

Problem 1: Let X and Y be metric spaces, let $f : X \to Y$ be a function, and let $\Gamma(f) = \{(x, y) \in X \times Y : f(x) = y\}$ be the graph of f. Show that if $\Gamma(f)$ is closed and Y is compact, then f is continuous.

Solution 1: Let $x_1, x_2, \cdots$ be a sequence in X converging to a point x. Let $y_1 = f(x_1), y_2 = f(x_2)$, and so on.

The compactness of Y implies that every subsequence of $y_1, y_2, \cdots$ has a subsequence that converges.

I now claim that for any pair of convergent subsequences $y_{n_1}, y_{n_2}, \cdots$ and $y_{m_1}, y_{m_2}, \cdots$ of $y_1, y_2, \cdots$, the limits $\lim_i y_{n_i}$ and $\lim_i y_{m_i}$ are the same. This is because the graph of f is closed and f is a function.

That is: $(x_{n_1}, y_{n_1}), (x_{n_2}, y_{n_2}), \cdots$ is a sequence in $\Gamma(f)$ that converges to $(x, \lim_i y_{n_i})$. Since $\Gamma(f)$ is closed it contains $(x, \lim_i y_{n_i})$; that is, $f(x) = \lim_i y_{m_i}$. Likewise, $(x_{m_1}, y_{m_1}), (x_{m_2}, y_{m_2}), \cdots$ is a sequence in $\Gamma(f)$ that converges to $(x, \lim_i y_{m_i})$. Since $\Gamma(f)$ is closed it contains $(x, \lim_i y_{m_i})$; that is $f(x) = \lim_i y_{m_i}$. Therefore $\lim_i y_{n_i} = \lim_i y_{n_i}$.

By Problem 1 on Problem set 6 the sequence $y_1, y_2, \cdots$ converges. And it must be that it converges to $f(x)$ because any subsequence of it does. This shows that f is continuous.

Problem 2: Show that the subset $\mathbf{Q} \subseteq \mathbf{R}$ of rational numbers is not a G_δ-set (that is, it cannot be obtained as a countable intersection of open sets).

Solution 2: Let $r_1, r_2, \cdots$ be an enumeration of the rational numbers. The subset $\mathbf{Q} - \mathbf{R}$ of irrational numbers is a G_δ set because it can be written as $\bigcap_{n=1}^{\infty} \mathbf{R} - \{r_n\}$ which is the countable intersection of open (and dense) subsets of $\mathbf{R}$. If the rational numbers were a G_δ set then they could be written as the intersection $\bigcap_{i=1}^{\infty} U_i$ where each U_i is open. Here, each U_i is dense because it contains the rational numbers. But then the countable intersection of open dense sets $\bigcap_{i=1}^{\infty} U_i \cap \bigcap_{n=1}^{\infty} \mathbf{R} - \{r_n\}$ is empty. This contradicts the Baire category theorem.

Problem 3: Let V be a Banach space. Show that the dimension of V is either finite or uncountable (that is, V does not have a countable basis).

Solution 3: Suppose that V had a countable basis $v_1, v_2, \cdots$. Define V_n to be the subspace of V spanned by the vectors $v_1, v_2, \cdots, v_n$. Each V_n is closed because it is a finite dimensional subspace of V. I claim that each V_n is nowhere dense. If U were an open subset of V contained in U then for each point v in U there would exist a number $\epsilon > 0$ such that $B_\epsilon(v)$ was a subset of U. To show that V_n is nowhere dense I will show that for no v in $V_n$and for no $\epsilon > 0$ is it the case that $B_\epsilon(v)$ is a subset of V_n. So let v be a point in V_n and let $\epsilon > 0$. Consider the point $v + tv_{n+1}$. This point does not belong to V_n for any $t > 0$ because $v_1, v_2, \cdots, v_{n+1}$ are linearly independent points. But we can choose t small enough so that this point belongs to $B_\epsilon(v)$. We can do this because $\|v + tv_{n+1} - v\|_V = \|tv_{n+1}\| = t\|v_{n+1}\|$. Therefore $B_\epsilon(v)$ is not a subset of V_n. This shows that V_n is nowhere dense. But we have that $V = V_1 \cup V_2 \cup \cdots$. This contradicts the Baire category theorem.

Problem 4: Let $E \subseteq \mathbf{R}^n$ be a measurable set with $0 < \mu(E) < \infty$. Let us regard $L^1(E)$ as a metric space, and $L^2(E)$ as a subset of $L^1(E)$. Show that $L^2(E)$ is meagre (that is, it is a countable union of nowhere dense subsets of $L^1(E)$).

Solution: Let $U_n = \{f \in L^1(E) : \int_E f^2 \leq n\}$. The set $L^2(E)$ is equal to $U_1 \cup U_2 \cup \cdots$. I will show that each U_n is nowhere dense by showing that U_n is closed and that $L^1(E) - U_n$ is dense in $L^1(E)$. Let f belong to $L_1(E) - U_n$. Then $\int_E f^2 > n$. Let $\epsilon > 0$. Let g be an element of $L^1(E)$ such that $\|f - g\|_{L^1} < \epsilon$. That is, $\int_E |f - g| < \epsilon$. What can we say about $\int_E (f-g)^2$? We know that $\int f^2 > n$. We have that

Let $f_1, f_2, \cdots$ be a sequence in U_n converging to f. Let g_i denote the function f_i^2 and let g denote the function f^2. Consider the set of points where g_i is larger than k and denote this set by $S_{i,k}$. What is the measure of this set? Since $\int_E g_i \leq n$ we have $k\mu(S_{i,k}) \leq n$ so that $\mu(S_{i,k}) \leq \frac{n}{k}$. Let $g_{i,k}$ denote the function defined by the equation $g_{i,k}(x) = g_i(x)\chi_{E-S_{i,k}}(x) + k\chi_{S_{i,k}}(x)$.

By continuity, as i tends to infinity this sequence of functions $g_{k,1}, g_{k,2}, \cdots$ converges to the function h_k defined by $h_k = g(x)\chi_{E-S_k}(x) + k\chi_{S_k}(x)$ where S_k is the set of points where g is larger than k.

Each term in the sequence $g_{1,k}, g_{2,k}, \cdots$ is measurable and bounded by k. Thus by the dominated convergence theorem the sequence of numbers $\int_E g_{1,k}, \int_E g_{2,k}, \cdots$ converges to $\int h_k$. Since each number in the sequence $\int_E g_{1,k}, \int_E g_{2,k}, \cdots$ is at most n the limit $\int h_k$ is at most n.

Now consider the sequence of functions $h_1, h_2, \cdots$. This is a nondecreasing sequence of functions which converges to g. By the monotone convergence theorem the sequence of numbers $\int_E h_1, \int_E h_2, \cdots$ converges to $\int_E g$. Since each number in the sequence $\int_E h_1, \int_E h_2, \cdots$ is at most n the limit $\int_E g$ is at most n.

This shows that f belongs to U_n. Therefore U_n is a closed set.

I will now show that $L^1(E) - U_n$ is dense in $L^1(E)$. Let f be an element of $L^1(E)$. We will show that there is a sequence of functions $f_1, f_2, \cdots$ in $L^1(E) - U_n$ which converges to f in the L^1 norm.

If f belongs to $L^1(E) - U_n$ we are done so suppose this is not the case. Therefore $\int_E f^2 \leq n$.

Each member of the sequence $f_1, f_2, \cdots$ satisfies $\int f_i^2 > n$. We want to choose the terms of this sequence so that the sequence of numbers $\int_E |f_1 - f|, \int_E |f_2 - f|, \cdots$ converges to 0.

Let $f_i = f + g_i$. Then the problem is to choose functions g_i in $L^1(E)$ so that the sequence of numbers $\int_E |g_1|, \int_E |g_2|, \cdots$ converges to 0 and such that $\int (f + g_i)^2 > n$.