**Jacob Lurie's 114. TFs: Stephen Mackereth, Patrick Ryan. Solution set 7.
Aubrey Clark**

Problem 1: Let $V_0, V_1, \ldots$ be real vector spaces with norms $\|\cdot\|_n : V_n \to \mathbf{R}_{\geq 0}$. Given an element $v = (v_n)_{n\geq 0} \in \prod_{n\geq 0} V_n$, let $\|v\| = \sum_{n\geq 0} \|v_n\|_n$. Let $V \subseteq \prod_{n\geq 0} V_n$ be the subset consisting of those elements v such that $\|v\| < \infty$. Show that V is a real vector space and that $v \mapsto \|v\|$ is a norm on V. If each V_n is a Banach space, show that V is a Banach space. We will refer to V as the l^1-sum of the Banach spaces $\{V_n\}_{n\geq 0}$.

Solution 1: We have to show that

1. V is a vector space,

2. $\|\cdot\| : V \to \mathbf{R}_{\geq 0}$ is a norm, and that

3. V with this norm is complete.

Let's show 1. The zero element of V is the vector 0_V whose n'th coordinate is the zero element of V_n. That is, if $v \in V$, then $v + 0_V = v$. Let $u, v \in V$. Then $u + v$ is the vector whose n'th coordinate is $u_n + v_n$. This belongs to V because

$$\|u+v\| = \sum_{n\geq 0} \|u_n+v_n\|_n \leq \lim_k \sum_{n=0}^{k} \|u_n\|_n + \|v_n\|_n = \lim_k \sum_{n=0}^{k} \|u_n\|_n + \lim_k \sum_{n=0}^{k} \|v_n\|_n = \|u\|_n + \|v\|_n < \infty.$$

Let $\alpha \in \mathbf{R}$ and let $v \in V$. We have that αv (the vector whose n'th coordinate is αv_n) belongs to V because

$$\|\alpha v\| = \lim_k \sum_{n=0}^{k} \|\alpha v_n\|_n = \sum_{n=0}^{k} |\alpha| \|v_n\|_n = |\alpha| \sum_{n\geq 0} \|v_n\|_n = |\alpha| \|v\|_n < \infty.$$

Therefore V is a vector space.

Let's show 2. We first have to show that $\|v\| = 0$ if and only if $v = 0_V$. If $v = 0_V$ then the n'th coordinate of v is the zero vector 0_{V_n} of V_n. Since each $\|\cdot\|_n$ is a norm we have that $\|v\| = \sum_{n\geq 0} \|0_{V_n}\|_n = 0$. Suppose that $\|v\| = 0$. Then $\sum_{n\geq 0} \|v_n\|_n = 0$. Each term in this sum is nonnegative. Therefore $\|v_n\|_n = 0$ and this implies (since $\|\cdot\|_n$ is a norm) that $v_n = 0_{V_n}$. It follows that $v = 0_V$.

Now let $\alpha \in \mathbf{R}$ and let $v \in V$. Then αv is the element of V whose n'th coordinate is αv_n. Then

$$\|\alpha v\| = \sum_{n\geq 0} \|\alpha v_n\|_n = \sum_{n\geq 0} |\alpha| \|v\|_n = |\alpha| \sum_{n\geq 0} \|v\|_n = |\alpha| \|v\|.$$

Finally, let $u, v \in V$. We have that

1

$$\|u+v\| = \sum_{n\geq 0}\|u_n+v_n\|_n \leq \lim_k \sum_{n=0}^{k}\|u_n\|_n + \|v_n\|_n = \lim_k \sum_{n=0}^{k}\|u_n\|_n + \lim_k \sum_{n=0}^{k}\|v_n\|_n = \|u\|_n + \|v\|_n.$$

.

Therefore $\|\cdot\|$ is a norm.

Let's show 3. Let $v_1, v_2, \ldots$ be a Cauchy sequence in V. That is, for each $\epsilon > 0$ there exists an integer N such that $\|v_i - v_j\| < \epsilon$ whenever $i, j \geq N$. This implies that each sequence $v_{1,n}, v_{2,n}, \cdots$ (where $v_{i,n}$ denotes the n'th coordinate of the vector v_i) is a Cauchy sequence in V_n. Since each V_n is a Banach space each of these sequences converges to some $u_n \in V_n$. I claim that the sequence $v_1, v_2, \ldots$ converges to the vector u whose n'th coordinate is u_n.

Let f_i be defined for each $x \geq 0$ by $f_i(x) = \|v_{i,\lfloor x\rfloor} - u_{\lfloor x\rfloor}\|_{\lfloor x\rfloor}$. Then $\|v_i - u\| = \int f_i \chi_{[0,\infty)}$. The sequence of functions $f_1, f_2, \cdots$ converges pointwise to zero. Since $v_1, v_2, \ldots$ is a Cauchy sequence in V it is bounded. Therefore the sequence $v_1 - u, v_2 - u, \cdots$ is bounded by some number M. This implies that for each i we have $|f_i| \leq M$. By The Bounded Convergence Theorem

$$\lim_i \|v_i - u\| = \lim_i \int f_i \chi_{[0,\infty)} = \int 0 \chi_{[0,\infty)} = 0.$$

Problem 2: Suppose we are given a sequence $E_0, E_1, \ldots \subseteq \mathbf{R}^m$ of pairwise disjoint measurable subsets of $\mathbf{R}^m$. Let $E = \bigcup E_n$. Show that $L^1(E)$ is isomorphic to the l^1-sum of the Banach spaces $L^1(E_n)$.

Solution 2: Associate with each element f of $L^1(E)$ the element $(f|_{E_1}, f|_{E_2}, \cdots)$. This defines a bijection between $L^1(E)$ and the l^1-sum of the Banach spaces $L^1(E_n)$. Denote by g_i the function $\sum_{n=1}^{i}|f|_{E_n}|$. Then the sequence $g_1, g_2, \cdots$ converges pointwise to $|f|$. We have that $|f|$ is integrable and $g_i \leq |f|$. By The Dominated Convergence Theorem

$$\lim \int g_i = \int |f|.$$

The right hand side of this expression is $\|f\|_{L^1}$. The left hand side can be written as

$$\sum_{n\geq 0}\|f|_{E_n}\|_{L^1}$$

which is the l^1-sum norm of $(f|_{E_1}, f|_{E_2}, \cdots)$.

Problem 3: Let $E \subseteq \mathbf{R}^n$ be a measurable set, let p and q be real numbers satisfying $\frac{1}{p} + \frac{1}{q} = 1$, and suppose that $f \in L^p(E)$, $g \in L^q(E)$ are functions satisfying

$$\int_E fg = \|f\|_{L^p}\|g\|_{L^q}.$$

Prove that either $f = 0$, or there exists a nonnegative real number λ such that $|g| = \lambda|f|^{\frac{p}{q}}$

almost everywhere.

Solution 3: By Young's inequality if x and y are positive numbers, then

$$xy \leq \frac{1}{p}x^p + \frac{1}{q}x^q.$$

This holds with equality if and only if $p \log(x) = q \log(y)$. That is, if $y = x^{\frac{p}{q}}$.

The equality holds if $f = 0$ so suppose f is not zero. Let $\hat{f} = \frac{f}{\|f\|_{L^p}}$ and $\hat{g} = \frac{g}{\|g\|_{L^q}}$. Then we have that $\int_E \hat{f}\hat{g} = 1$. We can write this as

$$\int_E |\hat{f}\hat{g}| = \frac{1}{p}\int_E |\hat{f}|^p + \frac{1}{q}\int_E |\hat{g}|^q.$$

By Young's inequality we have

$$|\hat{f}\hat{g}| \leq \frac{1}{p}|\hat{f}|^p + \frac{1}{q}|\hat{g}|^q$$

.

So by Problem 1 on Problem set 4 we have $|\hat{f}\hat{g}| = \frac{1}{p}|\hat{f}|^p + \frac{1}{q}|\hat{g}|^q$ almost everywhere. This implies that $|\hat{g}| = |\hat{f}|^{\frac{p}{q}}$ and this can be written as

$$|g| = \lambda|f|^{\frac{p}{q}}$$

where λ denotes the nonnegative real number $\frac{\|g\|_{L^q}}{\|f\|_{L^p}}$.

Problem 4: Let $p, q, r > 1$ be real numbers satisfying $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Let $E \subseteq \mathbf{R}^n$ be measurable, and let $f \in L^p(E)$ and $g \in L^q(E)$. Show that the product function fg belongs to $L^r(E)$, and that

$$\|fg\|_{L^r} \leq \|f\|_{L^p}\|g\|_{L^q}.$$

Solution 4: Let $p' = \frac{p}{r}$ and $q' = \frac{q}{r}$. Let $\hat{f} = f^r$ and $\hat{g} = g^r$. Holder's inequality tells us that

$$\|\hat{f}\hat{g}\|_{L^1} \leq \|\hat{f}\|_{L^{p'}}\|\hat{g}\|_{L^{q'}}.$$

We can rewrite this as

$$\left(\int_E |fg|^r\right) \leq \left(\int_E |f|^p\right)^{\frac{r}{p}}\left(\int_E |g|^q\right)^{\frac{r}{q}}.$$

And raising each side to the power of $\frac{1}{r}$ gives

$$\|fg\|_{L^r} \leq \|f\|_{L^p}\|g\|_{L^q}.$$

Since the right hand side is finite the inequality implies fg belongs to $L^r(E)$.