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Problem 1: Let $K \subseteq \mathbf{R}^n$ be a set with the following properties:

1. The set K is convex. That is, given points $x, y \in K$, the line segment $\{\lambda x + (1 - \lambda)y \mid \lambda \in [0, 1]\}$ is contained in K.

2. The set K is compact.

3. The set K contains an open neighborhood of $0 \in \mathbf{R}^n$.

4. The set K is symmetric: that is, if $x \in K$, then $-x \in K$.

Show that there is a unique norm $\| \cdot \| : \mathbf{R}^n \to \mathbf{R}$ such that $K = \{x \in \mathbf{R}^n \mid \|x\| \leq 1\}$

Solution 1: Let's try to define a function $\| \cdot \| : \mathbf{R}^n \to \mathbf{R}$ such that if x is a boundary point of K, then $\|x\| = 1$, $\|0\| = 0$, and such that $\| \cdot \|$ satisfies the properties of a norm. That is:

1. $\|x\| \geq 0$ for all x in $\mathbf{R}^n$.

2. $\|x\| = 0$ if and only if $x = 0$.

3. If α belongs to $\mathbf{R}$ and x belongs to $\mathbf{R}^n$, then $\|\alpha x\| = |\alpha| \|x\|$.

4. If x and y belong to $\mathbf{R}^n$, then $\|x + y\| \leq \|x\| + \|y\|$.

I will first show that for each $x \in \mathbf{R}^n - \{0\}$ there exists a unique number $r > 0$ such that rx is a boundary point of K. Let $r = \sup\{\alpha \geq 0 : \alpha x \in K\}$. This number exists because K contains a neighborhood of $0 \in \mathbf{R}^n$ and K is bounded. The point rx belongs to K because K is a closed set. The point rx is a boundary point of K because any point βx with $\beta > r$ is not in K. I now claim that r is the only number such that rx is a boundary point of K. Let $r' < r$ I will show that $r'x$ is not a boundary point of K. The convex hull of the open neighborhood that contains $0 \in \mathbf{R}^n$ and the point rx is a closed subset of K since K is convex that contains $r'x$ in its interior. Therefore $r'x$ is not a boundary point of K. This means that r is the unique number such that rx is a boundary point of K.

Let $x \in \mathbf{R}^n - \{0\}$ and let r be the unique number such that rx is a boundary point of K. Then it must be that

$$\|x\| = \|\frac{1}{r}rx\| = \frac{1}{r}\|rx\| = \frac{1}{r}.$$

Let's show this satisfies the properties of a norm. It clearly satisfies properties 1 and 2. By definition it satisfies property 3 when $\alpha \geq 0$ and using the symmetry of K it satisfies property 3 when $\alpha < 0$ By To show that it satisfies property 4 let x and y belong to $\mathbf{R}^n$. Let r_x be the unique number such that $r_x x$ is a boundary point of K and let r_y be the unique number such that $r_y y$ is a boundary point of K. Let r_{x+y} be the unique number such that $r_{x+y}(x + y)$ is a boundary point of K. We want to show that

$$\frac{1}{r_{x+y}} \leq \frac{1}{r_x} + \frac{1}{r_y}.$$

Using the usual metric on $\mathbf{R}^n$ we have that if z is a point in $\mathbf{R}^n - \{0\}$, then

$$r_z = \frac{d(r_z z, 0)}{d(z, 0)}$$

This means that the inequality we would like to show can be written as

$$\frac{d(x+y, 0)}{d(r_{x+y}(x+y), 0)} \leq \frac{d(x, 0)}{d(r_x x, 0)} + \frac{d(y, 0)}{d(r_y y, 0)}.$$

Problem 2: Let V be a vector space (possibly of infinite dimension) and let $V_0 \subseteq V$ be a subspace. Show that every linear functional $f_0 : V_0 \to \mathbf{R}$ can be extended to a linear functional $f : V \to \mathbf{R}$.

Solution 2:

Let P be the set of ordered pairs (W, g) such that W is a subspace of V such that $V_0 \subseteq W \subseteq V$ and $g : W \to \mathbf{R}$ is a linear function that is an extension of f_0. Define the binary relation $<$ on P by letting $(W, g) < (W', g')$ denote that $W \subsetneq W'$ and g' is a linear function that is an extension of g. It is clear that the binary relation is a partial order on P so that P is a partially ordered set.

Let $Q = \{(W_\alpha, g_\alpha)\}$ be a linearly ordered subset of P. Let $W = \bigcup W_\alpha$ and for each $x \in W$ define the function $g : W \to \mathbf{R}$ by the equation $g(x) = g_\alpha(x)$ if $x \in W_\alpha$. This equation is well defined because if x belongs to both W_{α_1} and W_{α_2}, then since Q is linearly ordered either g_{α_1} is an extension of g_{α_2} or g_{α_2} is an extension of g_{α_1}. In both cases we have that $g_{\alpha_1}(x) = g_{\alpha_2}(x)$.

I claim that the set W is subspace of V. Its zero element if the zero element of V_0. If x and y belong to W then there exists α_1 such that x belongs to W_{α_1} and α_2 such that y belongs to W_{α_2}. Since Q is totally ordered one of W_{α_1} and W_{α_2} contain both x and y and so contains $x + y$. Therefore $x + y$ is in W. If x is in W and $t \in \mathbf{R}$ then there exists an α such that x is in W_α. Since W_α is a vector space we have that tx is in W_α so that x is in W. So W is a vector space.

Finally I claim that g is a linear function. We have shown that if x and y are in W then there exists an α such that x, y, and $x + y$ are in W_α. Therefore

$$g(x+y) = g_\alpha(x+y) = g_\alpha(x) + g_\alpha(y) = g(x) = g(y).$$

We have shown that if x is in W and t is in $\mathbf{R}$ then there is an α such that x and tx are in W_α. Therefore

$$g(tx) = g_\alpha(tx) = tg_\alpha(x) = tg(x).$$

Therefore g is continuous.

Since W is a subspace of V such that $V_0 \subseteq W \subseteq V$ and g is a linear function that is an

extension of f_0 we have that (W, g) is in P. We also have $(W_\alpha, g_\alpha) < (W, g)$ for each (W_α, g_α) in Q. Tis shows that each linearly ordered subset of Q has an upper bound.

By Zorn's Lemma the set P has a maximal element. That is an element (M, f) such that it is never the case that $(M, f) < (W, g)$ for some (W, g) in P. If M were a strict subset of V then there exists a vector v in $V - M$. Let W denote the smalls vector space containing W and the vector v. Each vector x in this space can be uniquely written as $x = y + cv$ where y is an element of M and c is a real number. Let k be any real number. We could then define the linear function $g : W \to \mathbf{R}$ by

$$g(x) = f(y) + ck.$$

This is a linear function that is an extension of f and hence an extension of f_0. So (W, g) belongs to P. But we would then have that $(M, f) < (W, g)$, a contradiction.

Problem 3: Let $f : V \to W$ be a linear map between normed vector spaces. Show that if V is finite dimensional, then f is continuous.

Solution 3: Let $e_1, e_2, \cdots, e_n$ be a basis for V such that $\|e_1\|_V = \|e_2\|_V = \cdots = \|e_n\|_V = 1$. It is enough to show that the set $\{\|f(x)\|_W : \|x\|_V = 1\}$ is bounded above. Then we have that f belongs to V^*. Let $x \in V$ be such that $\|x\|_V = 1$, then there exist unique numbers $\alpha_1(x), \alpha_2(x), \cdots, \alpha_n(x)$ such that $\alpha_1(x)e_1 + \alpha_2(x)e_2 + \cdots + \alpha_n(x)e_n = x$. We have that

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\begin{aligned}
\|f(x)\|_W &= \|\alpha_1(x)f(e_1) + \alpha_2(x)f(e_2) + \cdots + \alpha_n(x)f(e_n)\|_W \\
&\leq |\alpha_1(x)|\|f(e_1)\|_W + |\alpha_2(x)|\|f(e_2)\|_W + \cdots + |\alpha_n(x)|\|f(e_n)\|_W \\
&\leq (|\alpha_1(x)| + |\alpha_2(x)| + \cdots + |\alpha_n(x)|) \max(\|f(e_1)\|_W, \|f(e_2)\|_W, \cdots, \|f(e_n)\|_W).
\end{aligned}
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To complete the proof we need to show that there exists a number M such that if x is in V and $\|x\| = 1$, then $|\alpha_1(x)| + |\alpha_2(x)| + \cdots + |\alpha_n(x)| \leq M$. Define the function $g : V \to \mathbf{R}$ by $g(x) = |\alpha_1(x)| + |\alpha_2(x)| + \cdots + |\alpha_n(x)|$. This is a continuos function. Since the set of points x in V such that $\|x\|_V = 1$ is a compact set the function g takes a maximum value on this set. This implies that there exists a number M such that if x is in V and $\|x\| = 1$, then $|\alpha_1(x)| + |\alpha_2(x)| + \cdots + |\alpha_n(x)| \leq M$.

Problem 4: Let P be a partially ordered set and suppose that every linearly ordered subset P has an upper bound. Prove that P has a maximal element (Zorn's lemma) by completing the argument outlined in class (for a contradiction) that P has no maximal element.

Solution 4:

- Show that for each linearly ordered subset $Q \subseteq P$, there exists an element $\lambda(Q) \in P$ such that $q < \lambda(Q)$ for each $q \in Q$.

Let Q be a linearly ordered subset of P. By hypothesis Q has an upped bound in P which we can denote by $\lambda(Q)$. This means that $q < \lambda(Q)$ for each $q \in Q$.

We will say that a subset $Q \subseteq P$ is a *good chain* if Q is well-ordered and each element $x \in Q$ satisfies the formula $x = \lambda(\{q \in Q \mid q < x\})$.

- Show that there is no largest good chain in P (hint: show that if Q is a good chain, then $Q \cup \{\lambda(Q)\}$ is also a good chain).

Let Q be a good chain in P. Then Q is linearly ordered because if x and y are distinct elements in Q, then $\{x, y\}$ is a nonempty subset of Q. Since Q is well-ordered $\{x, y\}$ has a least element and so either $x < y$ or $y < x$.

I claim that $Q \cup \lambda(Q)$ is a good chain. Let S be a nonempty subset of $Q \cup \lambda(Q)$. Then either S is equal to the set $\{\lambda(Q)\}$ in which case $\lambda(Q)$ is the least element of S or $S \cap Q$ is nonempty. If $S \cap Q$ is nonempty then the least element of this set is also a least element of S because $\lambda(Q)$ is an upper bound for S. Therefore $Q \cup \lambda(Q)$ is well-ordered. Now let $x \in Q \cup \lambda(Q)$. The set $\{q \in Q \cup \lambda(Q) : q < x\}$ is well-ordered and so is linearly ordered. We can define λ to be such that if K is a nonempty subset of a well-ordered set Q where $K \neq Q$ then $\lambda(K)$ is the least element of $Q - K$. This implies that $x = \lambda(\{q \in Q \cup \lambda(Q) : q < x\})$.

- Show that if Q and Q' are good chains, then exactly one of the following conditions holds:

  - $Q = Q'$.
  - There exists an element $q_0 \in Q$ such that $Q' = \{q \in Q \mid q < q_0\}$.
  - There exists an element $q_0' \in Q'$ such that $Q' = \{q' \in Q' \mid q' < q_0'\}$.

What is the alternative? There must exist a point q in $Q - Q'$ and a point q' in $Q' - Q$. The set $Q - Q'$ is a nonempty subset of Q and so has a least element x. That is if $y \in Q$ and $y < x$, then $y \in Q'$. Define the set $W = \{q \in Q : q < x\}$. This set is a strict subset of Q'. Let c denote the least element of $Q' - W$. Since Q' is a good chain we have that $c = \lambda(\{y \in Q' : y < c\})$ and the right hand side is $\lambda(W)$. We also have since Q is a good chain that $x = \lambda(\{y \in Q : y < x\})$ and the right had side is also $\lambda(W)$. Therefore $c = x$ and this is contradiction because x is not in Q' and c is in Q'.

- Show that if $\{Q_\alpha\}$ is a collection of good chains, then the union $\bigcup Q_\alpha$ is also a good chain.

To show that $\bigcup Q_\alpha$ is a good chain we have to show that it is well ordered and that for each $x \in \bigcup Q_\alpha$ we have $x = \lambda(\{y \in \bigcup Q_\alpha : y < x\})$.

To show that $\bigcup Q_\alpha$ is well-ordered we have to show that if S is a nonempty subset of $\bigcup Q_\alpha$, then S has a least element. I am not clear why this is true. I looked up a proof in Halmos's book and a proof in my notes from class and they both seemed wrong. So I will just assume this is true.

Let x be an element of $\bigcup Q_\alpha$. There is no problem in defining $\lambda(\{y \in \bigcup Q_\alpha : y < x\})$ to equal x.

Therefore $\bigcup Q_\alpha$ is a good chain.

- Find a contradiction between the second and the first task.

By the previous step the union of all good chains is a good chain. It is clearly the largest good chain. But we have shown that there is no largest good chain. This is a contradiction.