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Problem 1: Let V be a Banach space, and let $f : V \to \mathbf{R}^n$ be a linear map. Show that f is bounded if and only if the kernel $\ker(f)$ is a closed subset of V.

Solution 1: Suppose that f is bounded. Then f is continuous. The kernel $\ker(f)$ is closed because it is equal to the pre-image of a closed set under a continuous function.

Suppose that f is not bounded. Then there exists a sequence $v_1, v_2, \cdots$ in V such that $\|v_1\|_V = \|v_2\|_V = \cdots = 1$ and such that $\|f(v_1)\|_{\mathbf{R}^n}, \|f(v_2)\|_{\mathbf{R}^n}, \cdots$ diverges to infinity. Since $f \neq 0$ there exists a vector u such that $f(u) = 1$. Consider the sequence $x_1, x_2, \cdots$ whose ith term is defined by the equation $x_i = u - \frac{v_i}{f(v_i)}$. We have that $f(x_i) = f(u) - \frac{1}{f(v_i)} f(v_i) = 0$. That is, $x_1, x_2, \cdots$ is a sequence in $\ker(f)$. The sequence $x_1, x_2, \cdots$ converges to u since $\|u - x_i\|_V = |\frac{1}{f(v_i)}| \|v_i\|_V = |\frac{1}{f(v_i)}|$. Because u does not belong to $\ker(f)$ this implies that $\ker(f)$ is not closed.

Problem 2: Let V be a Banach space with norm $\|\cdot\|_V$, let $V_0 \subseteq V$ be a subspace, and let W denote the quotient V/V_0. Define a map $\|\cdot\|_W : W \to \mathbf{R}$ by the formula

$$\|x\|_W = \inf\{\|\tilde{x}\|_V : x \text{ represents } \tilde{x}\}.$$

Show that if V_0 is closed in V, then $\|\cdot\|_W$ is a norm which makes W into a Banach space.

Solution 2: The points of W represent a partition of the points of V. Two points x and y in V belong to the same member of this partition if there exists a point v_0 in V_0 such that $x = v_0 + y$. One way to think about this partition is the following: if x is a point in V then it belongs to the member of the partition which is equal to the points $\{x\} + V_0$.

To show that the function $\|\cdot\|$ defined above is a norm let's show that it has the properties of a norm.

Non-negativity: $\|x\| \geq 0$ *for all* $x \in W$. The non-negativity of $\|\cdot\|_V$ implies that the set $\{\|\tilde{x}\| : x \text{ represents } \tilde{x}\}$ is bounded below by zero. It follows that $\|\cdot\|$ is non-negative.

Zero-condition: $\|x\| = 0$ *if and only if* $x = 0_W$. The zero element of W corresponds to the subset V_0 of V. Since V_0 contains 0_V it follows that if $x = 0_W$, then $\|x\| = 0$. If $x \neq 0_W$ then the subset of V corresponding to x does not contain 0_V. Since V_0 is closed the set $\{x\} + V_0$ is closed. This means that there does not exists a sequence of points $x_1, x_2, \cdots$ in $\{x\} + V_0$ converging to 0_V. This implies that there exists a positive number ϵ such that $\|y\| \geq \epsilon$ whenever y belongs to $\{x\} + V_0$. Therefore $\|x\| \geq \epsilon$.

Linearity: $\|\alpha x\| = |\alpha| \|x\|$ *for all* $x \in W$ *and* $\alpha \in \mathbf{R}$. This follows from the zero condition if $\alpha = 0$ so suppose that $\alpha \neq 0$. The subset of V corresponding to x is the set $\{x\} + V_0$. The subset of V corresponding to αx is $\{\alpha x\} + V_0$. Note that $\alpha(\{x\} + V_0) = \{\alpha x\} + V_0$ since $\alpha V_0 = V_0$. Using this and that $\|\cdot\|_V$ is a norm we get

$$
\begin{aligned}
|\alpha|\|x\| =& |\alpha| \inf\{\|y\|_V : y \in \{x\} + V_0\} \\
=& \inf\{\|\alpha y\|_V : y \in \{x\} + V_0\} \\
=& \inf\{\|z\|_V : \frac{1}{\alpha} z \in \{x\} + V_0\} \\
=& \inf\{\|z\|_V : z \in \{\alpha x\} + V_0\} \\
=& \|\alpha x\|.
\end{aligned}
$$

Triangle inequality: if x_1 and x_2 belong to W, then $\|x_1 + x_2\| \leq \|x_1\| + \|x_2\|$. We have that $\|x_1 + x_2\| = \inf\{\|y_1 + y_2\| : y_1 + y_2 \in \{x_1 + x_2\} + V_0\}$ If $y_1 \in \{x_1\} + V_0$ and $y_2 \in \{x_2\} + V_0$, then $y_1 + y_2 \in \{x_1 + x_2\} + V_0$. Therefore

$$
\begin{aligned}
\|x_1 + x_2\| \leq& \inf\{\|y_1 + y_2\|_V : y_1 \in \{x_1\} + V_0 \text{ and } y_2 \in \{x_2\} + V_0\} \\
\leq& \inf\{\|y_1\|_V + \|y_2\|_V : y_1 \in \{x_1\} + V_0 \text{ and } y_2 \in \{x_2\} + V_0\} \\
=& \|x_1\| + \|x_2\|.
\end{aligned}
$$

So $\| \cdot \|$ is a norm on the set W. Therefore W with this norm is a normed space.

After unsuccessfully trying to prove that W is complete I came across the following useful result.

Lemma: A normed space X is a Banach space if and only if for all sequences $x_1, x_2, \cdots$ convergence of the series $\|x_1\| + \|x_2\| + \cdots$ implies that the series $x_1 + x_2 + \cdots$ converges in X.

Proof: Let X be a Banach space. Let $x_1, x_2, \cdots$ be a sequence in X such that $\|x_1\| + \|x_2\| + \cdots$ converges. Let $\epsilon > 0$. Choose N such that $\|x_N\| + \|x_{N+1}\| + \cdots < \epsilon$. Suppose that $n, m \geq N$. Without loss of generalness suppose that $n > m$. Then

$$
\begin{aligned}
\|\sum_{i=m}^{\infty} x_i - \sum_{i=n}^{\infty} x_i\| =& \|\sum_{i=m}^{n-1} x_i\| \\
\leq& \sum_{i=m}^{n-1} \|x_i\| \\
\leq& \sum_{i=m}^{\infty} \|x_i\| < \epsilon.
\end{aligned}
$$

Therefore $\{\sum_{i=1}^{n} x_i : n \in \mathbf{N}\}$ is a Cauchy sequence in X. Since X is a Banach space it converges in X.

Now suppose that for all sequences $x_1, x_2, \cdots$ convergence of $\|x_1\| + \|x_2\| + \cdots$ implies that the series $x_1 + x_2 + \cdots$ converges in X. Let $x_1, x_2, \cdots$ be a Cauchy sequence in X. There exists an integer N_1 such that $n, m \geq N_1$ implies that $\|x_n - x_m\| < \frac{1}{2^1}$. There exists an integer $N_2 > N_1$

such that $n, m \geq N_2$ implies that $\|x_n - x_m\| < \frac{1}{2^2}$. There exists an integer $N_3 > N_2$ such that $n, m \geq N_3$ implies that $\|x_n - x_m\| \leq \frac{1}{2^3}$. And so on. Now choose integers $n_1, n_2, \cdots$ such that $n_i \geq N_i$. Then $\|x_{n_k} - x_{n_{k+1}}\| < \frac{1}{2^k}$ which implies that the series $\|x_{n_1} - x_{n_2}\| + \|x_{n_2} - x_{n_3}\| + \cdots$ converges. By assumption this implies that the series $x_{n_1} - x_{n_2} + x_{n_2} - x_{n_3} + \cdots$ converges in X. Therefore the sequence $x_{n_1}, x_{n_2}, \cdots$ converges in X to some point x. Since $x_1, x_2, \cdots$ is a Cauchy sequence it also converges to x. So X is complete. Therefore X is a Banach space. QED

Now let $x_1, x_2, \cdots$ be a sequence in V/V_0 such that $\|x_1\| + \|x_2\| + \cdots$ converges. For each x_i there exists a $y_i \in \{x_i\} + V_0$ such that $\|y_i\|_V \leq 2\|x_i\|$. Then $\|y_1\|_V + \|y_2\|_V + \cdots$ converges. Since V is a Banach space the lemma implies that $y_1 + y_2 + \cdots$ converges in V to some point y. Therefore $x_1 + x_2 + \cdots$ converges in V/V_0 to $\{y\} + V_0$. Therefore V/V_0 is a Banach space.

Problem 3: Let $E \subseteq \mathbf{R}^n$ be a measurable set. Let $M(E)$ be the collection of all finite signed measures on E. For ν in $M(E)$ define

$$\|\nu\| = \sup\{\nu(S) - \nu(T)\}$$

where the supremum is taken over all pairs of disjoint measurable subsets $S, T \subseteq E$ such that $\nu(S) \geq 0$ and $\nu(T) \leq 0$. Show that the construction $\nu \mapsto \|\nu\|$ is a norm on the vector space $M(E)$ which makes $M(E)$ into a Banach space.

Solution 3: We need to show that $\|\cdot\|$ is a norm for $M(E)$ and that $M(E)$ with this norm is complete. Clearly $\|\cdot\|$ is nonnegative because the set of numbers $\{\nu(S) - \nu(T)\}$ is bounded below by zero (we can take S and T to be the empty sets). If ν is the finite signed measure that takes the value of zero on each subset of E, then $\nu(S) - \nu(T) = 0$ for all subsets S, T of E. Therefore $\|\nu\| = 0$. Now suppose that $\|\nu\| = 0$. Then it must be that ν is the finite signed measure that takes the value of zero on each subset of E (if there existed a subset of E on which ν took nonzero measure then we could make $\nu(S) - \nu(T)$ positive and then we would have $\|\nu\| > 0$). Now let ν_1, ν_2 belong to $M(E)$. We have that

$$\begin{aligned}
\|\nu_1 + \nu_2\| &= \sup\{(\nu_1 + \nu_2)(S) - (\nu_1 + \nu_2)(T)\} \\
&= \sup\{\nu_1(S) - \nu_1(T) + \nu_2(S) - \nu_2(T)\} \\
&\leq \sup\{\nu_1(S) - \nu_2(T)\} + \sup\{\nu_2(S) - \nu_2(T)\} \\
&= \|\nu_1\| + \|\nu_2\|.
\end{aligned}$$

So $\|\cdot\|$ is a norm on the set $M(E)$. To show that $M(E)$ with this norm is complete we need to show that if $\nu_1, \nu_2, \cdots$ is a Cauchy sequence in $M(E)$ then there exists a finite signed measure ν in $M(E)$ to which this sequence converges.

What does it mean for $\nu_1, \nu_2, \cdots$ to be a Cauchy sequence? It means that for each $\epsilon > 0$ there exists an integer N such that $\|\nu_n - \nu_m\| < \epsilon$ whenever $n, m \geq N$. That is, $\sup\{(\nu_n - \nu_m)(S) - (\nu_n - \nu_m)(T)\} < \epsilon$. We can rewrite this as $\sup\{(\nu_n(S) - \nu_n(T)) - (\nu_m(S) - \nu_m(T))\} < \epsilon$.

Let K be a subset of E. Define the following sequence of sets: $S_n = K$ and $T_n = \emptyset$ if $\nu_n(K) \geq 0$ and $S_n = \emptyset$ and $T_n = K$ if $\nu_n(K) < 0$. We have that

$$|(\nu_n(S_n) - \nu_n(T_n)) - (\nu_m(S_m) - \nu_m(T_m))| < \epsilon$$

whenever $n, m \geq N$. Therefore the sequence of numbers $\nu_1(S_1) - \nu_1(T_1), \nu_2(S_2) - \nu_2(T_2), \cdots$ is a Cauchy sequence of real numbers. Since the real numbers are complete this sequence converges to a number t. Define $\nu(K) = t$. This shows that the sequence of finite signed measure $\nu_1, \nu_2, \cdots$ conferges uniformly to ν. That is, from the above we have that if $n \geq N$, then $|\nu_n(K) - \nu(K)| < \epsilon$. This let's us interchange the supremum and the limit to show that if $n \geq N$, then

$$\sup\{(\nu_n(S) - \nu_n(T)) - (\nu(S) - \nu(T))\} < \epsilon.$$

That is, $n \geq N$ implies that $\|\nu_n - \nu\| < \epsilon$. This shows that $M(E)$ with this norm is complete. Therefore $M(E)$ with this norm is a Banach space.

Problem 4: Let $E \subseteq \mathbf{R}^n$ be a measurable set. Given a function $f \in L^1(E)$, define ν_f by the formula $\nu_f(S) = \int f\chi_S$. Show that ν_f is a finite signed measure on E, and that the construction

$$f \mapsto \nu_f$$

determines an isometry (that is, a norm-preserving map) from $L^1(E)$ onto a closed subspace of $M(E)$.

Solution 4: To show that ν_f is a finite signed measure we have to show that there exists a number C such that $-C \leq \nu(S) \leq C$ whenever S is a measurable subset of E and that ν_f is countably additive (if $S_1, S_2, \cdots$ are disjoint measurable subsets of E then $\nu_f(S_1 \cup S_2 \cup \cdots) = \nu_f(S_1) + \nu_f(S_2) + \cdots$).

If S is a measurable subset of E then since $-|f| \leq f\chi_S \leq |f|$ we have by monotonicity of the integral that $-\int |f| \leq \int f\chi_S \leq \int |f|$. That is, $-\int |f| \leq \nu_f(S) \leq \int |f|$. Therefore setting C equal to $\int |f|$ makes ν_f satisfy the first condition.

Next, let $S_1, S_2, \cdots$ be disjoint measurable subsets of E. We want to show that $\nu_f\left(\bigcup_{i=1}^\infty S_i\right) = \lim_n \sum_{i=1}^n \nu_f(S_i)$. By the linearity of the Lebesgue integral the right hand side of this expression can be written as $\lim_n \int f\chi_{\bigcup_{i=1}^n S_i}$. For each n $f\chi_{\bigcup_{i=1}^n S_i} \leq |f|$ and $\int |f| < \infty$ so by The Dominated Convergence Theorem $\lim_n \int f\chi_{\bigcup_{i=1}^n S_i} = \int \lim_n f\chi_{\bigcup_{i=1}^n S_i}$ and the right hand side of this expression is equal to $\int f\chi_{\bigcup_{i=1}^\infty S_i}$ which is equal to $\nu_f\left(\bigcup_{i=1}^\infty S_i\right)$. This shows that ν_f is countably additive. Therefore ν_f is a finite signed measure.

To show that the map $f \mapsto \nu_f$ is an isometry we need to show that for each function f in $L^1(E)$ we have that $\|f\|_{L^1} = \|\nu_f\|$ where the norm used on the right hand side of this equation is the norm from problem 3.

This equation can be written as $\int |f| = \sup\{\nu_f(S) - \nu_f(T) : \nu_f(S) \geq 0 \text{ and } \nu_f(T) \leq 0\}$.

We can write $\nu_f(S) - \nu_f(T)$ as $\int f(\chi_S - \chi_T)$. By monotonicity of the integral the supremum on the right is achieved when S is taken to be the set of points where f is nonnegative and T is taken to be the complement of S in E. Then we have that $\nu_f(S) - \nu_f(T) = \int |f|$ so that $\|f\|_{L^1} = \|f\|$. This shows that the map $f \mapsto \nu_f$ is an isometry.