Jacob Lurie's 114. TFs: Stephen Mackereth, Patrick Ryan. Solution set 5. Aubrey Clark

Problem 1: Let $E \subseteq \mathbf{R}^n$ be a measurable set, and let $f_0 \leq f_1 \leq f_2 \leq \cdots$ be an increasing sequence of integrable functions on E for which the sequence of integrals $\left\{\int_E f_i\right\}_{i\geq 0}$ is bounded. Show that the sequence $\{f_i\}$ converges almost everywhere to an integrable function f, and that $\int_E f$ is a limit of the sequence $\left\{\int_E f_i\right\}_{i\geq 0}$.

Solution 1: If $\{f_i\}$ converges almost everywhere to an integrable function f, then by the dominated convergence theorem $\int_E f$ is the limit of the sequence $\left\{\int_E f_i\right\}_{i\geq 0}$. The sequence $\left\{\int_E f_i\right\}_{i\geq 0}$ is increasing and bounded so it converges to some real number.. If $\{f_i(x)\}_{i\geq 0}$ does not converge then it is unbounded above. So if the set $\{x \in E \mid \{f_i(x)\}_{i\geq 0}$ does not converge $\}$ has positive measure, then the sequence $\left\{\int_E f_i\right\}_{i\geq 0}$ could not be bounded above. Therefore the sequence $\{f_i\}_{i\geq 0}$ converges almost everywhere to a function f. This function is integrable becasue $\left\{\int_E f_i\right\}_{i\geq 0}$ is bounded.

Problem 2: Let $f : \mathbf{R} \to \mathbf{R}$ be an integrable function. Show that $\int f$ is a limit of the sequence of real numbers $\left\{\int f \mid_{[-n,n]}\right\}_{n\geq 0}$

Solution 2: Denote $f \mid_{[-n,n]}$ by g_n. Since the function $\mid f \mid$ is integrable, $\mid g_n \mid \leq \mid f \mid$, and the sequence $\{g_n\}$ converges pointwise to f the dominated convergence theorem implies that $\int f$ is a limit of the sequence of real numbers $\left\{\int f \mid_{[-n,n]}\right\}_{n\geq 0}$.

Problem 3: Let $f : \mathbf{R}^n \to \mathbf{R}$ be an integrable function, and suppose that $\int f \mid_B = 0$ for every open box $B \subseteq \mathbf{R}^n$. Prove that f vanishes almost everywhere.

Solution 3: We can write $f = f_+ - f_-$ where $f_+(x)$ is equal to $f(x)$ when $f(x) > 0$ and 0 othewise and $f_-(x)$ is equal to $-f(x)$ when $f(x) < 0$ and 0 otherwise. Then

$$0 = \int f \mid_B = \int f_+ \mid_B - \int f_- \mid_B$$

so that $\int f_+ \mid_B = \int f_- \mid_B$ for every open box $B \subseteq \mathbf{R}^n$. Now suppose that f does not vanish on a set $S \subseteq \mathbf{R}^n$ with positive measure. Then $S \cap \{x \mid f(x) > 0\}$ or $S \cap \{x \mid f(x) < 0\}$ has positive measure. Whichever of these has positive measure we can find an open box B which is a subset of it. Then it can't possibly be true that $\int f_+ \mid_B = \int f_- \mid_B$.

Problem 4: Let E be the subset of $[0, 1]$ consisting of those real numbers whose decimal expansion contains infinitely many occurrences of the digit 7. Show that E is a measurable set, and compute its measure.

Solution 4: Consider the complement of E in $[0, 1]$. This is the set of elements of $[0, 1]$ whose decimal expansion has finitely many 7s. Let A_n be the set of elements in $[0, 1]$ whose decimal expansion does not contain any 7s after the n'th digit. Then $[0, 1] - E = A_1 \cup A_2 \cup \cdots$. Note that the outer measure of each A_n is zero (i.e. $1^n 0.9^\infty$). Countable subadditivity of the outer measure implies that $[0, 1] - E$ is a set of measure zero and so is a measurable set. The set E is measurable because of the closure properties of measurable sets and has measure 1 by the countable additivity of Lebesgue measure.