Jacob Lurie's 114. TFs: Stephen Mackereth, Patrick Ryan. Solution set 6. Aubrey Clark

Problem 1: Let X be a metric space, and let $\{x_n\}_{n\geq 0}$ be a sequence of points in X which satisfies the following conditions:

1. For every subsequence $\{x_{i_0}, x_{i_1}, \cdots\}$, there exists a further subsequence $\{x_{i_{j_0}}, x_{i_{j_1}}, \cdots\}$ which converges.

2. For any pair of convergent subsequences $\{x_{i_0}, x_{i_1}, \cdots\}, \{x_{j_0}, x_{j_1}, \cdots\}$ of $\{x_n\}_{n\geq 0}$, the limits $\lim\{x_{i_n}\}$ and $\lim\{x_{j_n}\}$ are the same.

Show that the sequence $\{x_n\}_{n\geq 0}$ converges.

Solution 1: By condition 1, let $\{x_{i_0}, x_{i_1}, \cdots\}$ be a convergent subsequence of $\{x_n\}_{n\geq 0}$ with limit x. Then, since $\{x_n\}_{n\geq 0}$ does not converge, there exists a subsequence of $\{x_n\}_{n\geq 0}$ such that each point in this subsequence is at least distance r from x. By condition 1 this subsequence has a further subsequence which converges. But the limit of this further subsequence is at least distance r from x. This contradicts condition 2.

Problem 2: Let E be a measurable subset of $\mathbf{R}^m \times \mathbf{R}^n$. For each $x \in \mathbf{R}^m$, let $E_x = \{y \in \mathbf{R}^n \mid (x, y) \in E\}$. Show that E has measure zero if and only if the sets E_x have measure zero for almost every x.

Solution 2:

I will use the notation introduced in class. Suppose E has measure zero. Let n be a positive integer. There exists open boxes $B_1, B_2, \cdots$ whose union contains E and such that $\mu(B_1) + \mu(B_2) + \cdots \leq \frac{1}{n}$. Let F_n denote $B_1 \cup B_2 \cup \cdots$. For each n we have that $E_x \subset F_{n,x}$ and so $\mu(E_x) \leq \mu(F_{n,x})$. We have that $\int f_{F_n} = \mu(F) \leq \frac{1}{n}$. Let S_k denote the set of points where f_{F_n} takes a values larger than $\frac{1}{k}$. The measure of S_k is no more than $\frac{k}{n}$ (otherwise $int f_{F_n}$ would exceed $\frac{1}{n}$). This implies that the measure of the set where f_E has value larger than $\frac{1}{k}$ is at most $\frac{k}{n}$. Since this is true for any k and any n it follows that f_E is zero almost everywhere. That is, the sets E_x have measure zero for almost every x.

Now suppose that E_x has measure zero for almost every x. We proved in class that

1. E_x is measurable for almost every $x \in \mathbf{R}^m$,

2. $f_E : \mathbf{R}^m \to [0, \infty]$ defied by $f_E(x) = \mu(E_x)$ is a measurable function, and

3. $\int f_E = \mu(E)$.

Since the integral of the zero function is equal to zero we have $\mu(E) = 0$ from (3).

Problem 3: Let $f : \mathbf{R}^2 \to \mathbf{R}$ be the function given by

$$f(x,y) = \begin{cases} 1 & \text{if } (\exists n \in \mathbf{Z}_{\geq 0})[n \leq x, y < n+1], \\ -1 & \text{if } (\exists n \in \mathbf{Z}_{\geq 0})[n \leq x < n+1 \leq y < n+2], \\ 0 & \text{otherwise} \end{cases}$$

For each $x \in \mathbf{R}$, let f_x denote the function given by $f_x(y) = f(x, y)$. For each $y \in \mathbf{R}$, let f_y denote the function given by $f_y(x) = f(x, y)$. Show that the functions

$$x \mapsto \int f_x \; y \mapsto \int f_y$$

are integrable, and compute their integrals. Why does this result not contradict Fubini's theorem?

Solution 3:

We have

$$\int f_x = \begin{cases} 0 & \text{if } x < -1, \\ -1 & \text{if } -1 \leq x < 0, \\ 0 & \text{if } 0 \leq x. \end{cases}$$

and

$$\int f_y = \begin{cases} 0 & \text{if } y < 0, \\ 1 & \text{if } 0 \leq y < 1, \\ 0 & \text{if } 1 \leq y. \end{cases}$$

Clearly both these functions are integrable. We have $\int (\int f_x) = -1$ and $\int (\int f_y) = 1$. This does not contradict Fubini's theorem because f is not an integrable function. That is, $\int |f| = \infty$.

Problem 4: Let $\phi : \mathbf{R} \to \mathbf{R}$ be a continuous function satisfying the inequality

$$\phi\left(\frac{x+y}{2}\right) \leq \frac{\phi(x) + \phi(y)}{2}$$

for all $x, y \in \mathbf{R}$. Show that ϕ is convex: that is, for each real number $\lambda \in [0, 1]$, we have

$$\phi(\lambda x + (1 - \lambda)y) \leq \lambda\phi(x) + (1 - \lambda)\phi(y)$$

for all $x, y \in \mathbf{R}$.

Solution 4: Suppose ϕ is not convex but it is continuous and satisfies the given inequality. Then there exist points x, y in $\mathbf{R}$ and a number $\lambda \in [0, 1]$ such that

$$\phi(\lambda x + (1 - \lambda)y) > \lambda\phi(x) + (1 - \lambda)\phi(y)$$

Let's approximate λ. Let $k_n = \lfloor \lambda 2^n \rfloor$. Then for each positive integer n

$$\frac{k_n}{2^n} \leq \lambda \leq \frac{(k_n + 1)}{2^n} = \frac{k_n}{2^n} + \frac{1}{2^n}.$$

This implies that $\lim_n \frac{k_n}{2^n} = \lambda$. Since ϕ is continuous we have that

$$\lim_n \phi\left(\frac{k_n}{2^n}x + \frac{2^n - k_n}{2^n}y\right) > \lim_n \left(\frac{k_n}{2^n}\phi(x) + \frac{2^n - k_n}{2^n}\phi(y)\right).$$

So for n sufficiently large

$$\phi\left(\frac{k_n}{2^n}x + \frac{2^n - k_n}{2^n}y\right) > \frac{k_n}{2^n}\phi(x) + \frac{2^n - k_n}{2^n}\phi(y).$$

But by repeatedly applying the given inequality to the left hand side we get

$$\begin{aligned}\phi\left(\frac{k_n}{2^n}x + \frac{2^n - k_n}{2^n}y\right) &= \phi\left(\frac{x}{2^n} + \frac{k_n - 1}{2^n}x + \frac{2^n - k_n}{2^n}y\right) \\ &\leq \frac{\phi(x) + \phi\left(\frac{k_n-1}{2^{n-1}}x + \frac{2^n-k_n}{2^{n-1}}y\right)}{2} \\ &\dots \\ &\leq \frac{k_n}{2^n}\phi(x) + \frac{2^n - k_n}{2^n}\phi(y),\end{aligned}$$

a contradiction.