Decision Theory Lecture Notes[1]

Economics 2059, Harvard University

[1] Based on Tomasz Strzalecki's 2011-2012 lectures; created by Aubrey Clark, Divya Kirti, Annie Liang, and Gleb Romanyuk.

Lecture 1

Choice, Preferences, and Utility: Static Models

1.1 Introduction

Let X be a set of alternatives. Define as follows:

- A *choice function* $c : 2^X \setminus \{\emptyset\} \to 2^X$ such that $c(A) \subseteq A$.
- A *preference relation* given by the binary relation $\succ \subseteq X \times X$.
- A *utility function* $u : X \to \mathbb{R}$.

The exercises in decision theory include: 1. Finding conditions on a choice function such that it can equivalently be represented by a preference relation of some type.; 2. Searching for conditions on a preference relation such that it can equivalently be represented by a utility function of some type.

1.1.1 Structures on X

We want additional structure on X in order to properly describe the alternatives. Useful structures include:

1. A topology: (X, τ)
2. Risk: $X = \Delta(Z)$ or a mixture space
3. Commodity bundles, time: $X = Z^n$

4. Menus: $X = 2^Z \setminus \{\emptyset\}$

5. Random choice: $C : 2^X \setminus \{\emptyset\} \to \Delta(X)$ satisfying supp $C(A) \subseteq A$, e.g.

$$C(x) = \frac{u(x)}{\sum_{y \in A} u(y)}$$

1.1.2 Properties of binary relations

Let B be a binary relation on X. Then $\succ$ is:

1. *Reflexive* if $\forall\, x \in X, xBx$

2. *Irreflexive* if $\forall\, x \in X, \neg(xBx)$

3. *Symmetric* if $\forall\, x, y \in X, xBy \Rightarrow yBx$

4. *Asymmetric* if $\forall\, x, y \in X, xBy \Rightarrow \neg(yBx)$

5. *Antisymmetric* if $\forall\, x, y \in X, (xBy \wedge yBx) \Rightarrow x = y$

6. *Transitive* if $\forall\, x, y, z \in X, (xBy \wedge yBz) \Rightarrow xBz$

7. *Negatively transitive* if $\forall\, x, y, z \in X, \neg(xBy) \wedge \neg(yBz) \Rightarrow \neg(xBz)$

8. *Complete* if $\forall\, x, y, z \in X, (xBy) \vee (yBx)$

Let $\succ$ be a binary relation on X. We can interpret $x \succ y$ as the decision maker's strict preference of x to y, when faced with the pair of alternatives x and y.

Lemma 1.1.1. *If $\succ$ is asymmetric, then $\succ$ is irreflexive.*

Lemma 1.1.2. *If $\succ$ is asymmetric and negatively transitive, then $\succ$ is transitive.*

Definition $\succ$ is called a *weak order* if and only if it is asymmetric and negatively transitive.

Given the weak order $\succ$ let $\sim$ be the binary relation defined by

$$x \sim y \iff \neg(x \succ y) \wedge \neg(y \succ x)).$$

Additionally, let $\succeq$ be the binary relation defined by

$$x \succeq y \iff \neg(y \succ x).$$

These examples illustrate the derivation of $\sim$ from $\succ$.

Example 1.1.3. *Cars:* $X = \{Ferrari, Mclaren, Nissan\}$

$$\succ = \{(Ferrari, Mclaren), (Ferrari, Nissan), (Mclaren, Nissan)\},$$

$$\sim = \{(Ferrari, Ferrari), (Mclarin, Mclaren), (Nissan, Nissan)\}$$

Example 1.1.4. *Beer:* $X = \{1, 2, 3\}$

$$\succ = \{(1, 3)\},$$

$$\sim = \{(1, 2), (2, 1), (2, 3), (3, 2), (1, 1), (2, 2), (3, 3)\}$$

Lemma 1.1.5. *If $\succ$ is a weak order, then $\forall\, x, y \in X$ exactly one of the following statements is true:*

1. $x \succ y$.
2. $y \succ x$.
3. $x \sim y$.

Lemma 1.1.6. *If $\succ$ is a weak order, then*

1. $(x \sim y) \wedge (y \succ z) \Rightarrow x \succ z$.
2. $(x \succ y) \wedge (y \sim z) \Rightarrow x \succ z$.

1.2 Preference representation

Definition $\succ$ is *represented* by $u : X \to \mathbb{R}$ if and only if for all $x, y \in X$,

$$x \succ y \iff u(x) > u(y).$$

Theorem 1.2.1. *Suppose X is finite. Then $\succ$ is a weak order if and only if there exists $u : X \to \mathbb{R}$ that represents $\succ$.*

Theorem 1.2.2. *Suppose X is countable. Then $\succ$ is a weak order if and only if there exists $u : X \to \mathbb{R}$ that represents $\succ$.*

For the proof let $W(x) = \{z \in X \mid x \succ z\}$, enumerate X, and define

$$u(x) = \sum_{i | z_i \in W(x)} \frac{1}{2^i}.$$

The following examples illustrate that additional axioms are needed for representation if X is uncountable.

Example 1.2.3. *Suppose the cardinality of X is more than the continuum and that $\{z \in X \mid z \sim x\} = \{x\}$. There does not exist $u : X \to \mathbb{R}$ that represents $\succ$.*

Example 1.2.4. *Suppose $X = \mathbb{R} \times \{0, 1\}$ and $(a, n) \succ (b, m)$ if and only if $a \succ b$ or $(a = b) \wedge (n > m))$. There does not exist $u : X \to \mathbb{R}$ that represents $\succ$.*

Definition The set $Z \subseteq X$ is $\succ$*-order dense* iff for all $x, y \in X$ if $x \succ y$ there exists $z \in Z$ such that $x \succeq z \succeq y$.

Theorem 1.2.5. *Suppose $\succ$ is a weak order on X and $\exists\ Z \subseteq X$ countable and $\succ$-order dense if and only if $\exists u : X \to \mathbb{R}$ that represents $\succ$.*

For the proof let $W(x) = \{z \in Z \mid x \succ z\}$, $B(x) = \{z \in Z \mid z \succ x\}$, enumerate Z, and define

$$u(x) = \sum_{i | z_i \in W(x)} \frac{1}{2^i} - \sum_{i | z_i \in B(x)} \frac{1}{2^i}.$$

Proposition 1.2.6. *Suppose $\succ$ is a weak order represented by $u : X \to \mathbb{R}$. Then it is represented by $u' : X \to \mathbb{R}$ if and only if there exists $f : u(X) \to \mathbb{R}$, strictly increasing s.t. $u' = f \circ u$.*

For the proof define f as follows. For any $\zeta \in u(X)$ there exists an $x \in X$ such that $\zeta = u(x)$. Define $f(\zeta) = u'(x)$. Check that this f is correct and that it is strictly increasing.

1.3 Revealed preference theory

Let X be a finite set of alternatives and define $P(X) = 2^X \setminus \{\emptyset\}$. That is, $P(X)$ represents the set of potential choice sets the agent may face.

Definition A *choice function* for X is a function $c : P(X) \to P(X)$ such that for all $A \subseteq X$, $c(A) \subseteq A$.

Let $\succ$ be a binary relation on X. Define the function $c(\cdot, \succ) : P(X) \to P(X)$ by

$$c(A, \succ) = \{x \in A \mid \text{ for all } y \in A, \neg(y \succ x)\}.$$

Questions to consider:

1. Given $\succ$, when is $c(\cdot, \succ)$ a choice function?

2. Given a choice function, when is there a binary relation $\succ$ such that $c(\cdot) = c(\cdot, \succ)$?

Definition A choice function is *nonempty* if for all $A \in P(X)$, $c(A) \neq \emptyset$

Definition $\succ$ is *acyclic* if for all integers n and for all alternatives $x_1, x_2, ..., x_n$, if $x_1 \succ x_2 \succ \cdots \succ x_{n-1} \succ x_n$, then $x_1 \neq x_n$.

Theorem 1.3.1. *$c(\cdot, \succ)$ is nonempty if and only if $\succ$ is acyclic.*

Definition (Sen's α condition) If $x \in A \subseteq B$ and $x \in c(B)$, then $x \in c(A)$.

Theorem 1.3.2. *For any binary relation $\succ$, $c(\cdot, \succ)$ satisfies Sen's α condition.*

Example 1.3.3. *$X = \{a, b, c\}$, $\succ = \{(a, c)\}$. Then $c(\{a, b\}, \succ) = \{a, b\}$, $c(\{b, c\}, \succ) = \{b, c\}$, $c(\{a, c\}, \succ) = \{a\}$, $c(\{a, b, c\}, \succ) = \{a, b\}$.*

Definition (Sen's β condition) If $x, y \in A \subseteq B$, $x, y \in c(A)$, and $y \in c(B)$, then $x \in c(B)$.

Theorem 1.3.4. *If $\succ$ is a negatively transitive binary relation, then $c(\cdot, \succ)$ satisfies Sen's β condition.*

Theorem 1.3.5. *A choice function $c : P(X) \to P(X)$ satisfies Sen's α and β conditions if and only if there exists a weak order $\succ$ on X such that $c(A) = c(A, \succ)$ for all $A \in P(X)$.*

Lecture 2

Objective Probabilities

2.1 von Neumann-Morgenstern

Let Z be a set of alternatives and let $\Delta(Z)$ be the set of probability measures on Z. Let $\succ$ be a binary relation on $\Delta(Z)$.

Axiom 2.1.1. *Z is finite.*

Axiom 2.1.2. *$\succ$ is a weak order on $\Delta(Z)$.*

Axiom 2.1.3. Independence: *For all p,q,$r \in \Delta(Z)$ and $a \in [0,1]$, $p \succ q$ implies $ap+(1-a)r \succ aq+(1-a)r$.*

Axiom 2.1.4. Continuity: *For all p, q, $r \in \Delta(Z)$, if $p \succ q \succ r$, then there exist $a, b \in (0,1)$ such that $ap + (1-a)r \succ q \succ bp + (1-b)r$.*

Axiom 2.1.5. Calibration: *For all p, q, $r \in \Delta(Z)$, if $p \succeq q \succeq r$, then there exists $\alpha \in (0,1)$ such that $q \sim \alpha p + (1-\alpha)r$.*

Axiom 2.1.6. Mixture continuity: *For all p, q, $r \in \Delta(Z)$, if $p \succeq q \succeq r$, then the sets $\{\alpha \in [0,1] \mid \alpha p + (1-\alpha)r \succ q\}$ and $\{\alpha \in [0,1] \mid q \succ \alpha p + (1-\alpha)r\}$ are open in $[0,1]$.*

Axiom 2.1.7. *There exist r_1, $r_2 \in \Delta(Z)$ such that if $p \in \Delta(Z)$, then $r_1 \succeq p \succeq r_2$.*

Theorem 2.1.8. *The binary relation satisfies axioms 2.1, 2.1, 2.1, 2.1, 2.1, 2.1, 2.1, if and only if there exists a function $u : Z \to \mathbb{R}$ such that $p \succ q$ iff $\sum_{z \in Z} u(z)p(z) > \sum_{z \in Z} u(z)q(z)$.*

Lemma 2.1.9. *Let $U : \Delta(Z) \to \mathbb{R}$. Then U is linear if and only if there exists $u : Z \to \mathbb{R}$ such that $U(p) = \sum_{z \in Z} u(z)p(z)$.*

The theorem is proved by defining $U(p) = \alpha$ such that $p \sim \alpha r_1 + (1-\alpha)r_2$, showing U is linear, and then showing that U represents $\succ$.

Theorem 2.1.10. *If U and U' are linear and represent $\succ$ on ΔZ, then there exist real numbers $a > 0, b$ such that $U'(p) = aU(p) + b$ for all $p \in \Delta Z$.*

To obtain a representation theorem in the case where Z is infinite we introduce the idea of a mixture space.

Definition A *mixture space* is a set Π endowed with a function $\{h_\alpha\}_{\alpha \in [0,1]}$ such that $h_\alpha : \Pi^2 \to \Pi$ and for all $\pi, \rho \in \Pi$

1. $h_1(\pi, \rho) = \pi$.
2. $h_\alpha(\pi, \rho) = h_{1-\alpha}(\rho, \pi)$.
3. $h_\alpha(h_\beta(\pi, \rho), \rho)$.

Example 2.1.11. *$\Delta(Z)$ is a mixture space. $(\Delta(Z))^S$ is a mixture space.*

Theorem 2.1.12. *Let Π be a mixture space. $\succ$ satisfies,*

1. *$\succ$ on Π is a weak order,*
2. *$\pi \sim \rho$ implies $h_{\frac{1}{2}}(\pi, \rho) = h_{\frac{1}{2}}(\rho, \pi)$, and*
3. *Mixture continuity: For all π, ρ, μ in Π the sets $\{\alpha \in [0,1] \mid h_\alpha(\pi, \rho) \succ \mu\}$ and $\{\alpha \in [0,1] \mid \mu \succ h_\alpha(\pi, \rho)\}$ are open in $[0,1]$,*

if and only if there exists $U : \Pi \to \mathbb{R}$ linear such that U represents $\succ$. Moreover, if U and U' are linear functions from Π to $\mathbb{R}$ that both represent $\succ$ then $U' = aU + b$ for some constants $a > 0$ and b.

2.2 The Allais paradox

The utility function of Theorem 2.1 implies that indifference curves are parallel straight lines. This is also implied by Axiom 2.1.

Example 2.2.1. *(The Allais paradox.)*

$$a_1 = \left\{ \text{1.00 chance of \$1,000,000} \right. \qquad \text{versus} \qquad a_2 = \left\{ \begin{array}{l} \text{.10 chance of \$5,000,000} \\ \text{.89 chance of \$1,000,000} \\ \text{.01 chance of \$0} \end{array} \right.$$

and

$$a_3 = \left\{ \begin{array}{l} \text{.10 chance of \$5,000,000} \\ \text{.90 chance of \$0} \end{array} \right. \qquad \text{versus} \qquad a_4 = \left\{ \begin{array}{l} \text{.11 chance of \$1,000,000} \\ \text{.89 chance of \$0} \end{array} \right.$$

The choice of a_1 in the first pair and a_3 in the second pair implies that indifference curves are not parallel straight lines.

2.3 Dekel 86

Axiom 2.3.1. $\succ$ *is a weak order on* $\Delta(Z)$.

Axiom 2.3.2. *There exists a worst element* $\underline{z}$ *and a best element* $\bar{z}$ *in* Z.

Axiom 2.3.3. *Solvability: If* $P, Q, R \in \Delta(Z)$ *and* $P \succ Q \succ R$*, then there exists* $\alpha \in (0,1)$ *such that* $\alpha P + (1-\alpha)R \sim Q$.

Axiom 2.3.4. *Monotonicity: If* $z', z'' \in \Delta(Z)$ *such that* $z' \succ z''$ *[resp:* $z' \sim z''$*] and* $w = \underline{z}$ *or* $w = \bar{z}$*, then* $\alpha z' + (1-\alpha)w \succ \alpha z'' + (1-\alpha)w$ *[resp:* $\alpha z' + (1-\alpha)w \sim \alpha z'' + (1-\alpha)w$*] for every* $\alpha \in [0,1]$.

Axiom 2.3.5. *Betweenness: If* $P \succ Q$ *[resp:* $P \sim Q$*], then* $P \succ \alpha P + (1-\alpha)Q \succ Q$ *for all* $\alpha \in (0,1)$ *[resp:* $P \sim \alpha P + (1-\alpha)Q \sim Q$ *for all* $\alpha \in [0,1]$

The betweenness axiom says that indifference curves are linear.

Theorem 2.3.6. *Preferences over* $\Delta(Z)$ *satisfy axioms 2.3,2.3,2.3,2.3,2.3 if and only if there exists a function* $u : Z \times [0,1] \to \mathbb{R}$ *increasing in the preference ordering on* Z*, and continuous in the second argument such that* $P \succ Q$ *[resp:* $P \sim Q$*] if and only if* $V(P) > V(Q)$ *[resp:* $V(P) = V(Q)$*], where* $V(F)$ *is defined implicitly as the unique* $v \in [0,1]$ *that solves*

$$\int u(w,v)dF(w) = vu(\bar{z},v) + (1-v)u(\underline{z},v).$$

Furthermore $u(w,v)$ is unique up to positive affine transformations which are continuous functions of v. A particular transformation exists setting $u(\underline{z},v)=0$ and $u(\bar{z},v)=1$ for every v, giving the simpler representation (similar to expected utility)

$$\int u(w,V(F))dF(w)=V(F).$$

2.4 Gul, 91

This paper uses a special case of betweenness preferences. For some b,w such that $b>w$, let $X=[w,b]$ be the set of all prizes. Let $\mathcal{L}$ be the set of simple lotteries over these prizes (i.e. with finite support).

Axiom 2.4.1. *Preference relation: $\succeq$ is a complete and transitive binary relation on $\mathcal{L}$.*

Axiom 2.4.2. *Continuity: For all $p\in\mathcal{L}$ the sets $\{q\in\mathcal{L}\mid q\succeq p\}$ and $\{q\in\mathcal{L}\mid p\succeq q\}$ are closed (under the topology generated by the L^1 metric).*

Definition For any $\succeq$ and p, let

$$B(p)=\{q\in\mathcal{L}\mid z\in\text{supp}(q)\}\text{ implies }\delta_z\succeq q\}$$

$$W(p)=\{q\in\mathcal{L}\mid z\in\text{supp}(q)\}\text{ implies }q\succeq\delta_z\}$$

This p will be a reference point.

Definition (α,q,r) is called an *elation/disappointment decomposition* of p if $q\in B(p)$, $r\in W(p)$, and $\alpha q+(1-\alpha)r=p$.

Let $D(p)$ denote the set of all elation/disappointment decompositions of p.

Axiom 2.4.3. *Weak independence: $p_1\succeq p_2$, $a\in[0,1]$, $x\in X$ implies $ap_1+(1-a)z\succeq ap_2+(1-a)z$ whenever there exists $(\lambda,q_i,r_i)\in D(p_i)$ such that $q_i\in B(ap_i+(1-a)z)$ and $r_i\in W(ap_i+(1-a)z)$ for $i=1,2$.*

Axiom 2.4.4. *Symmetry: For $i=1,2$, (α,p,w), $(\alpha,b,p_i)\in D$ implies*

$$\alpha p_1+(1-\alpha)w\succeq\alpha p_2+(1-\alpha)w\iff\alpha b+(1-\alpha)p_1\succeq\alpha b+(1-\alpha)p_2.$$

Theorem 2.4.5. $\succeq$ *satisfies Axioms 2.4,2.4,2.4,2.4 if and only if there exist functions* $u : X \to \mathbb{R}$ *and* $\gamma : [0,1] \to [0,1]$ *such that*

1. $(\alpha_i, q_i, r_i) \in D(p_i)$ *for* $i = 1, 2$ *implies* $p_1 \succeq p_2$ *iff*

$$\gamma(\alpha_1) \sum_x u(x) q_1(x) + (1 - \gamma(\alpha_1)) \sum_x u(x) r_1(x) \geq \gamma(\alpha_2) \sum_x u(x) q_2(x) + (1 - \gamma(\alpha_2)) \sum_x u(x) r_2(x),$$

2. γ', u' *satisfy 1 above implies* $u' = au + b$ *for some* $a > 0$, $b \in \mathbb{R}$ *and* $\gamma' = \gamma$.

3. u *is continuous and there exists* $\beta \in (-1, \infty)$ *such that*

$$\gamma(\alpha) = \frac{\alpha}{1 + (1 - \alpha)\beta} \text{ for all } \alpha \in [0,1].$$

Lecture 3

Risk and Time

3.1 Preliminaries

Theorem 3.1.1 (Elienberg[1]). *Suppose a connected and separable topological space (X, τ). A binary relation $\succ$ on X is a continuous weak order if and only if there exists a function $u : X \to \mathbb{R}$ that is continuous in τ and represents $\succ$.*

3.2 Time

The main reference is Fishburn Ch. 4,5.

Assume binary relation $\succ$ is a weak order defined on $X_1 \times ... \times X_n$. The question of the section is: Which axioms have to be imposed so that we can write the utility function in the form $u(x) = u(x_1) + ... + u(x_n)$?

Definition Let $x, y \in X$, $E \subseteq \{1, ...n\}$. Define $x_E y \in X$ as

$$x_E y = \begin{cases} x_i, & i \in E \\ y_i, & i \notin E \end{cases}$$

The next definition is spiritually related to the independence axiom.

Definition Relation $\succ$ satisfies *separability* if $\forall x, y, z, z' \in X$ and $\forall E \subseteq \{1, ...n\}$ $x_E z \succ y_E z$ if and only if $x_E z' \succ y_E z'$. That is any set E is *independent*.

Definition Dimension i is *null* if for all x, y, z $x_i z \sim y_i z$.

[1] This is how it was called by Tomasz. In Fishburn, it is called after Debreu (p.62)

In other words, the agent does not care about dimension i.

Theorem 3.2.1 (Debreu). *Suppose: (1) each X_i for $i = 1, ...n$ is a connected separable topological space, (2) $X = X_1 \times ... \times X_n$, and (3) at least three components are non-null. A binary relation $\succ$ on X is (i) a weak order; (ii) continuous in product topology; (iii) satisfies separability if and only if there exist functions $u_i : X_i \to \mathbb{R}$, u_i is constant if i is null, such that*

$$x \mapsto \sum_{i=1}^{n} u_i(x_i)$$

represents $\succ$. Moreover, if $(u_1, ...u_n)$ and $(v_1, ...v_n)$ represent $\succ$, then there exist $\alpha > 0$, $\beta_1, ...\beta_n \in \mathbb{R}$ such that $v_i = \alpha u_i + \beta_i$.

Theorem 3.2.2 (Gorman). *Substitute separability in Theorem 3.2.1 with the following condition: $\{i, i+1\}$ are independent for $i = 1, ...n-1$.*

Theorem 3.2.3 (Debreu or Thomsen or Radomeister or hexagon or E_3). *Suppose $X = X_1 \times X_2$, X_i is a connected and separable topological space, $i = 1, 2$. Substitute separability in Theorem 3.2.1 with the following condition: $(a_1, b_2) \sim (b_1, a_2)$ and $(a_1, c_2) \sim (c_1, a_2) \sim (b_1, b_2)$ implies $(b_1, c_2) \sim (c_1, b_2)$.*

Theorem 3.2.4. *$X = X_1 \times X_2$. A binary relation $\succ$ on X is (i) a weak order; (ii) continuous in product topology; (iii) satisfies separability if and only if there exist functions $u_1 : X_1 \to \mathbb{R}$, $u_2 : X_2 \to \mathbb{R}$ and a function $G : u_1(X_1) \times u_2(X_2) \to \mathbb{R}$ strictly increasing in both components such that*

$$(x_1, x_2) \mapsto G(u_1(x_1), u_2(x_2))$$

represents $\succ$.

Proof. Necessity part is a simple check, we are proving sufficiency here.

Step 0. Invoke Elienberg's theorem (Theorem 3.1.1) to get $U(x_1, x_2)$.

Step 1. Fix $x_2^\circ \in X_2$. Define $u_1(x_1) = U(x_1, x_2^\circ)$. For all x_2 $u_1(x_1) > u_1(x_1')$ is equivalent to $U(x_1, x_2) > U(x_1', x_2)$. Function $u_1(\cdot)$ represents a preference $\succ^*$ on X_1 and due to separability $U(\cdot, x_2)$ represents the same preference $\succ^*$ on X_1. Therefore, by ordinal uniqueness, there exists a function $F(\cdot, x_2) : u(X_1) \to \mathbb{R}$ strictly increasing such that $U(x_1, x_2) = F(u(x_1), x_2)$.

Step 2. Fix $x_1^\circ \in X_1$ and define $u_2(x_2) = U(x_1^\circ, x_2)$. Again, by separability of $\succ$, for all $x_1 \in X_1$, $u_2(x_2) > u_2(x_2')$ iff $F(u_1(x_1), x_2) > F(u_1(x_1), x_2')$. By ordinal uniqueness, there exists a function $G(u_1(x_1), \cdot) : u_2(X_2) \to \mathbb{R}$ such that $G(u_1(x_1), u_2(x_2)) = F(u_1(x_1), x_2)$. QED. □

We will work with three domains:

- $X = C^T$, T is finite
- $X = C^{\mathbb{N}}$
- $X = C \times \mathbb{N}$ or $X = C \times \mathbb{R}$

The first two correspond to consumption streams, and the third is known as dated rewards. Note that the continuous time is missing here as well as infinite time. Popular preference representations include:

- $u(x_0, x_1, ...) = W(u(x_0), U(x_1, ...))$ - recursive.
- $\sum \delta(t)u(x_t)$ - weighted utilities.
- $\sum \delta^t u(x_t)$ - geometric discounting.
- $u(x_0) + \beta \sum_{t=1} \delta^t u(x_t)$ - quasi-hyperbolic discounting.

3.3 Fishburn Ch. 7.

The state space is $X = C^{T+1}$, C is a connected separable topological space, $T > 1$.

Axiom 3.3.1. $\succ$ *is a weak order.*

Axiom 3.3.2 (Sensitivity of the first period)**.** *There exist numbers $a, b \in C$, $x \in X$ such that $(a, x_1, ...x_T) \succ (b, x_1, ...x_T)$.*

Axiom 3.3.3 (Initial tradeoff independence)**.** *For all $a, b, c, d \in C$, $x, y \in X$ $(a, b, x_2, ...x_T) \succ (c, d, x_2, ...x_T)$ if and only if $(a, b, y_2, ...y_T) \succ (c, d, y_2, ...y_T)$.*

Axiom 3.3.4 (Stationarity)**.** *$(a, x_0, ...x_{T-1}) \succ (a, y_0, ...y_{T-1})$ if and only if $(x_0, ...x_{T-1}, a) \succ (y_0, ...y_{T-1}, a)$.*

Axiom 3.3.5. $\succ$ *is continuous in product topology.*

Theorem 3.3.6. *Let Axioms 1-5 hold. Then there exists a unique number $\delta > 0$, and a function $u : C \to \mathbb{R}$ continuous and unique up to a positive affine transformation such that*

$$x \longmapsto \sum_{t=0}^{T} \delta^t u(x_t)$$

represents $\succ$.

Proof. **Step 1**. Invoke Gorman's independence. Axiom 3 implies $\{0,1\}$ is independent. Let $(x_0,a,b,x) \succ (x_0,c,d,x)$; by Axiom 4, $(a,b,x,x_0) \succ (c,d,x,x_0)$; by Axiom 3, $(a,b,y,y_0) \succ (c,d,y,y_0)$; by Axiom 4 again $(y_0,a,b,y) \succ (y_0,c,d,y)$. Therefore, $\{1,2\}$ is independent. By induction, obtain that $\{t,t+1\}$ is independent. Further, Axiom 2 implies that $\{0\}$ is non-null. By applying Axiom 4 sufficient number of times, $\{t\}$ is non-null. By Gorman's theorem (Theorem 3.2.2), there exist functions $V_0, V_1, ...V_T : C \to \mathbb{R}$, continuous such that $x \mapsto \sum_{t=0}^{T} V_t(x_t)$ represents $\succ$ that is unique up to similar affine transformation.

Step 2. Fix $e \in C$ and define $\succ^*$ on C^T:

$$(x_0, x_1, ...x_{T-1}) \succ^* (y_0, y_1, ...y_{T-1}) \quad \text{iff} \quad (x_0, x_1, ...x_{T-1}, e) \succ (y_0, y_1, ...y_{T-1}, e)$$

Use Gorman representation from the last step for $\succ$ to get that $\succ^*$ can be represented by $(x_0, ...x_{T-1}) \mapsto \sum_{t=0}^{T-1} V_t(x_t)$. By Axiom 4, $(x_0, x_1, ...x_{T-1}, e) \succ (y_0, y_1, ...y_{T-1}, e)$ implies $(e, x_0, x_1, ...x_{T-1}) \succ (e, y_0, y_1, ...y_{T-1})$. Therefore, $\succ^*$ can be represented by $(x_0, ...x_{T-1}) \mapsto \sum_{t=0}^{T-1} V_{t+1}(x_t)$. By cardinal uniqueness of the representation, $\exists a > 0, b_0, ...b_{T-1} \in \mathbb{R}$ such that

$$V_{t+1}(c) = aV_t(c) + b_t$$

Define $w_0 = v_0$, $w_1 = aw_0$, $w_2 = aw_1$ and so on. So $(x_0, ...x_T) \mapsto \sum_{t=0}^{T} w_t(x_t)$ represents $\succ$. Set $u := w_0$ and $\delta := a$. QED. □

Remark 3.3.7. *In this theorem, δ is just positive and not bounded from above.*

3.4 Bleichrodt (2008)

C is a connected and separable topological space, $\mathcal{T} = \{0, 1, 2, ...\}$, $X = C^{\mathcal{T}}$. Define,

$$x_T y = \begin{cases} x_i, & i \leqslant T \\ y_i, & i > T \end{cases}$$

Definition x is called ultimately constant if $x = x_T c$ for some T and $c \in C$.

Denote by X_T all ultimately constant plans of length T. The space $\mathcal{F} \subseteq X$ contains all ultimately constant plans. Let $\succ$ be a relation defined on X.

The axioms used in the paper come in part from the previous section.

Axiom 3.4.1. *$\succ$ is a weak order.*

Axiom 3.4.2. *Sensitivity of the first period.*

Axiom 3.4.3. *Initial tradeoff independence.*

Axiom 3.4.4 (Stationarity)**.** *$x \succ y$ iff $cx \succ cy$ for all $x, y \in \mathcal{F}$ and for all $c \in C$.*

Axiom 3.4.5 (Constant equivalence)**.** *For all $x \in \mathcal{F}$ there exists $c \in C$ such that $c \sim x$.*

Axiom 3.4.6 (Finite continuity)**.** *For any T $\succ|_{X_T}$ is continuous in the product topology.*

Axiom 3.4.7 (Tail continuity)**.** *For all $x \in \mathcal{F}$ and for all $c \in C$ the following two implications hold: (i) $x \succ c$ implies that there exists τ such that for any $T > \tau$ $x_T c \succ c$; (ii) $c \succ x$ implies that there exists τ such that for any $T > \tau$ $c \succ x_T c$.*

Theorem 3.4.8. *Suppose Axioms 1-7 hold. Then there exist a number $\delta \in [0,1]$ and a function $u : C \to \mathbb{R}$ continuous, non-constant such that*

$$x \longmapsto \sum_{t=0}^{\infty} \delta^t u(x_t)$$

represents $\succ$ on $\mathcal{F}$.

Proof. **Step 1.** Axiom 3+Axiom 4 imply that $\{t, t+1\}$, $\{t, t+1, ...\}$ are independent for $t = 1, 2, ...$

Step 2. By Axiom 2 and Axiom 4, each $\{t\}$ is sensitive.

Step 3. Get the additive representation on X_T. Invoke Gorman's theorem: Axioms 1, 6 and Steps 1 and 2 imply that $\succ|_{X_T}$ is represented by,

$$(x_0, x_1, ..., x_T, c, ...) \mapsto \sum_{t=0}^{T} V_{t,T}(x_t) + R_T(c).$$

Fix $e \in C$. For all t normalize:

$$V_{t,T}(e) = R_T(e) = 0. \tag{3.1}$$

Step 4. $X_t \subseteq X_{T+1}$. Therefore $(x_0, x_1, ..., x_T, c, ...)$ can also be represented as

$$(x_0, x_1, ..., x_T, c, c...) \mapsto \sum_{t=0}^{T} V_{t,T+1}(x_t) + V_{T+1,T+1}(c) + R_{T+1}(c)$$

By cardinal uniqueness and (3.1), there exists $\gamma_{T+1} > 0$ such that

$$\begin{aligned} V_{t,T+1}(c) &= \gamma_{T+1} V_{t,T}(c) \\ V_{T+1,T+1}(c) + R_{T+1}(c) &= \gamma_{T+1} R_T(c) \end{aligned}$$

By uniqueness, choose another family of representations (essentially, divide $V_{t,T+1}$ by γ_{T+1}):

$$V_{t,T} = V_{t,T+1} =: V_t.$$

For all z:

$$V_{T+1}(z) + R_{T+1}(z) = R_T(z) \tag{3.2}$$

Step 5. By Axiom 4, get two representations:

$$\begin{aligned} (x_0, ...x_T, c, c, ...) &\mapsto \sum_{t=0}^{T} V_t(x_t) + R_T(c) \\ (x_0, ...x_T, c, c, ...) &\mapsto \sum_{t=0}^{T} V_{t+1}(x_t) + R_{T+1}(c) \end{aligned}$$

By uniqueness, there exists $\delta_T > 0$ such that for all z:

$$\begin{aligned} V_{t+1}(z) &= \delta_T V_t(z) \\ R_{T+1}(z) &= \delta_T R_T(z) \end{aligned} \tag{3.3}$$

Since V_t is independent of T, $\delta_T = \delta$.

Step 6. Set $u = V_0$ and $R = \frac{1}{\delta} R_0$. From (3.2) using (3.3): $\delta^T u(z) + \delta^{T+1} R(z) = \delta^T R(z)$. If $\delta = 1$, then u is constant, and that would be a contradiction. Therefore $\delta \neq 1$ and we can solve for $R(z) = \frac{1}{1-\delta} u(z)$. If $\delta > 1$, there exist $a, b \in C$ such that $u(a) > u(b)$.

$$(x_0, ...x_T, c) \mapsto \sum_{t=0}^{T} \delta^t u(x_t) + \frac{\delta^{t+1}}{1-\delta} u(c)$$

Therefore $u(b) + \delta u(b) + \frac{\delta^2}{1-\delta} u(b) > u(a) + \delta u(a) + \frac{\delta^2}{1-\delta} u(a)$ and $b \succ a$ (as constant streams). By Axiom 7, there exists T such that $b_T a \succ a$. Use representation: $(1 + \delta + ... + \delta^T) u(b) + \frac{\delta^{T+1}}{1-\delta} u(a) > (1 + \delta + ... + \delta^T) u(a) + \frac{\delta^{T+1}}{1-\delta} u(a)$, that gives $u(b) > u(a)$. A contradiction.

Step 7. Take some $x \in \mathcal{F}$. By Axiom 5, $\exists c \in C$ s.t. $c \sim x$. Consider two cases:

Case 1. There exists another constant $a \in C$ such that $c \succ a$.

Case 2. For all $a \in C$: $a \succsim c$.

Under Case 1, by Axiom 7, there exists a number τ such that for all $T > \tau$: $x_T a \succ a$. Or using the representation from Step 6:

$$\sum_{t=0}^{T} \delta^t u(x_t) + \frac{\delta^{t+1}}{1-\delta} u(a) > \sum_{t=0}^{T} \delta^t u(a) + \frac{\delta^{t+1}}{1-\delta} u(a)$$

$$\sum_{t=0}^{T} \delta^t [u(x_t) - u(a)] > 0$$

So, there exists τ such that $\inf_{T \geqslant \tau} \left\{ \sum_{t=0}^{T} \delta^t [u(x_t) - u(a)] \right\} \geqslant 0$. It can be rewritten as

$$\sup_{\tau} \inf_{T \geqslant \tau} \left\{ \sum_{t=0}^{T} \delta^t [u(x_t) - u(a)] \right\} \geqslant 0$$

$$\liminf_{T} \left\{ \sum_{t=0}^{T} \delta^t u(x_t) - \sum_{t=0}^{T} \delta^t u(a) \right\} \geqslant 0$$

$$\liminf_{T} \sum_{t=0}^{T} \delta^t u(x_t) \geqslant \sum_{t=0}^{\infty} \delta^t u(a)$$

Since this holds for all $a \prec c$ and $u(\cdot)$ is continuous, then

$$\liminf_{T} \sum_{t=0}^{T} \delta^t u(x_t) \geqslant \sum_{t=0}^{\infty} \delta^t u(c)$$

Now consider Case 2. For all T there exists b such that $x_T c \sim b$. So $x_T c \succsim c$ (since we are in case 2). For all T,

$$\sum_{t=0}^{T} \delta^t u(x_t) + \frac{\delta^{t+1}}{1-\delta} u(c) \geqslant \sum_{t=0}^{T} \delta^t u(c) + \frac{\delta^{t+1}}{1-\delta} u(c)$$

$$\liminf_{T} \sum_{t=0}^{T} \delta^t [u(x_t) - u(c)] \geqslant 0$$

Repeat Cases 1 and 2 with limsup's and get the equalities. QED. $\square$

3.5 Kreps, Porteus (1978)

Assume, given a finite integer T and for each time t ($t = 0, 1...., T$), a set Z_t, of possible payoffs. We assume that each Z_t, is a compact Polish (i.e., complete separable metric) space. A generic element of Z_t, is denoted by z_t. Let $Y_1 = Z_0$ and, for $t = 2, ..., T+1$, let $Y_t = Y_{t-1} \times Z_{t-1}$. The set Y_t is called the set of payoff *histories* up to (but not including) time t, with generic element $y_t = (z_0, ..., z_{t-1})$. Note that Y_{T+1} is the set of complete payoff vectors.

Next, let D_T be the set of Borel probability measures on Z_T, endowed with the Prohorov metric (the metric of weak convergence), $D_T = \Delta(Z_T)$, and, recursively, let X_t be the set of nonempty closed subsets of D_t, $X_t = K(D_t)$, endowed with the Hausdorff metric, and let $D_{t-1} = \Delta(Z_{t-1} \times X_t)$, endowed with the Prohorov metric. An element of X_t is the choice set at time t.

For notational convinience we will denote sigleton sets by their elements, e.g. $\delta_{(z_t,x_t)} =: (z_t, x_t)$.

Example 3.5.1. *$Z_0 = Z_1 = \{0, 1\}$, $T = 1$, $D_1 = \Delta(\{0, 1\})$ - a set of lotteries between 0 and 1. $X_1 = K(D_1)$, $D_0 = \Delta(\{0, 1\} \times K(\Delta(\{0, 1\})))$ - price today times the decision problem tomorrow. $X_0 = K(D_0)$. See Figure 3.1.*

Remark 3.5.2. *Each set D_t is a mixture space: For $\alpha \in [0, 1]$ and $d, d' \in D_t = \Delta(Z_t \times X_{t+1})$, there is an element in D_t which "is" d with probability α and d' with probability $1-\alpha$. Let $(\alpha; d, d')$ denote this element.*

Definition For each real valued bounded measurable function $f : Z \times X_{t+1} \to \mathbb{R}$ and for each $d \in D_t$, the integral of f with respect to the measure d is denoted by $E_d[f]$.

Axiom 3.5.3 (2.1 in KP). *For each t and y_t, $\succsim_{y_t}$ is a weak order on D_t.*

Axiom 3.5.4 (2.2 in KP). *For each y_t, $\succsim_{y_t}$ is continuous.*

Axiom 3.5.5 (2.3 in KP, independence). *For each y_t, if $d, d' \in D_t$ are such that $d \succ_{y_t} d'$, then that for all $\alpha \in (0, 1)$ and for all $d'' \in D_t$: $(\alpha; d, d'') \succ_{y_t} (\alpha; d', d'')$.*

Lemma 3.5.6 (Lemma 3 in KP). *Axioms 2.1, 2.2, and 2.3 are necessary and sufficient for there to exist, for each y_t, a (bounded) continuous function $U_{y_t} : Z_t \times X_{t+1} \to \mathbb{R}$ (bernoulli) such that for $d, d' \in D_t$, $d \succsim_{y_t} d'$ if and only if $E_d U_{y_t} \geqslant E_{d'} U_{y_t}$. Function U_{y_t} is unique up to a positive affine transformation.*

The function U_{y_t} can be extended to $D_t = \Delta(Z_t \times X_{t+1})$ by defining $U_{y_t}(d) = E_d[U_{y_t}]$. Thus, U_{y_t} is a representation of $\succsim_{y_t}$ on D_t, ($\succsim_{y_t}$ compares lotteries). It can be extended farther to X_t by defining

$$\hat{U}_{y_t}(x) = \max_{d \in x} U_{y_t}(d).$$

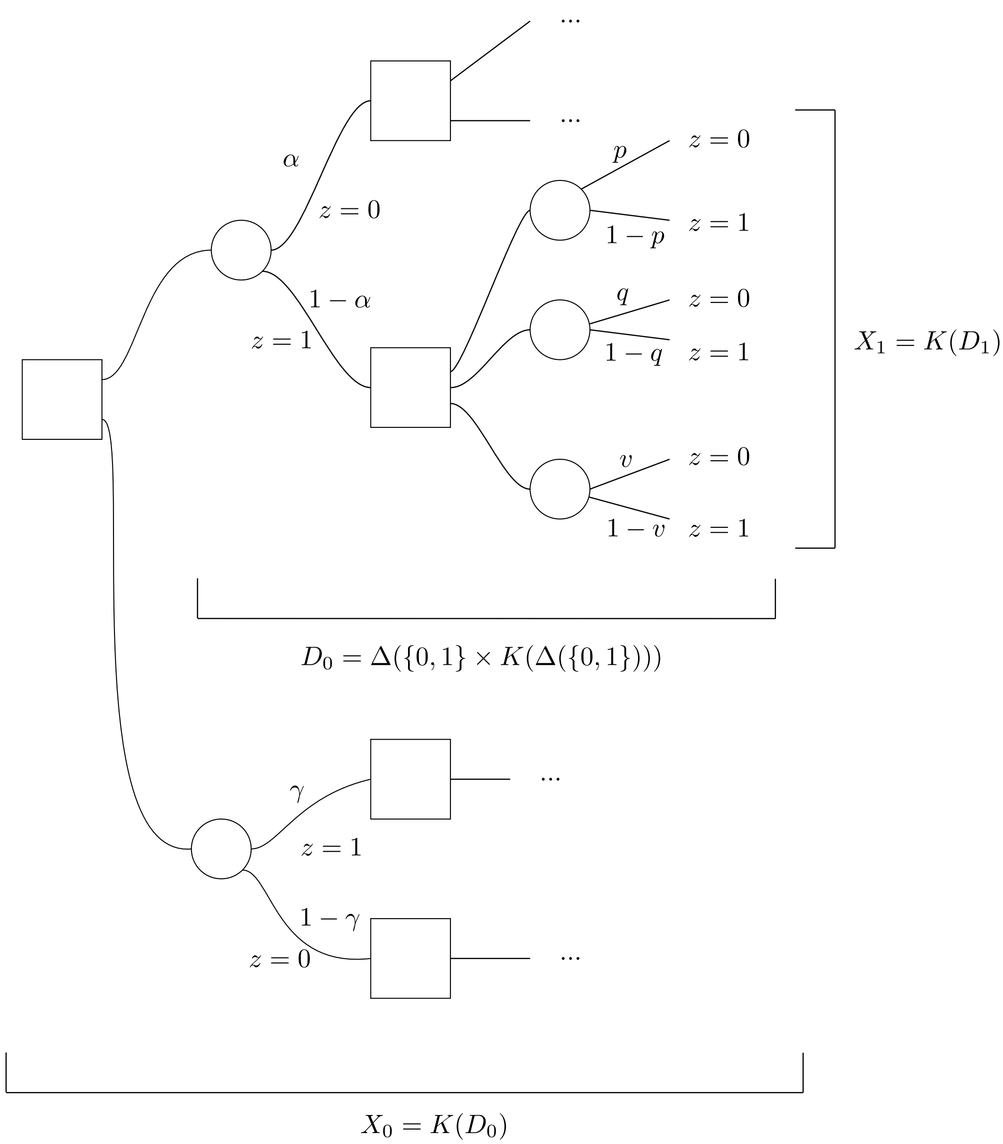


Figure 3.1: Decision tree. Square nodes represent decision nodes, circle nodes are "luck" nodes.

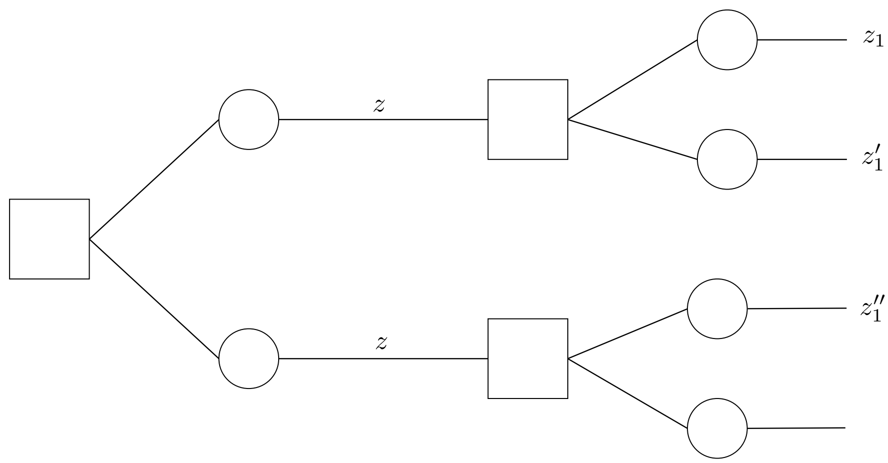


Figure 3.2: Illustration of dynamic consistency

The function $\hat{U}_{y_t}$ represents preference $\hat{\succsim}_{y_t}$ on X_t ($\hat{\succsim}_{y_t}$ compares sets of lotteries): $x \hat{\succsim}_{y_t} x'$ iff for each $d' \in x'$ there exists a $d \in x$ such that $d \succsim_{y_t} d'$, or alternatively, due to compactness of x and x', there exists a $d \in x$ such that for all $d' \in x'$, $d \succsim_{y_t} d'$.

Note that $\hat{\succsim}_{y_t}$ is complete, transitive and continuous.

Axiom 3.5.7 (3.1 in KP, dynamic/temporal consistency)**.** *For each $y_t \in Y_t$, $z \in Z_t$ and $x, x' \in X_{t+1}$*

$$(z, x) \succsim_{y_t} (z, x') \quad \textit{iff} \quad x \,\hat{\succsim}_{(y_t, z)}\, x'$$

See Figure 3.2.

Axiom 3.5.8 (3.1')**.** *For each y_t and z_t and for all $d, d' \in D_t$, $(z, d) \succsim_{y_t} (z, d')$ iff $d \succsim_{(y_t, z)} d'$.*

By induction we can get that if $y_t = (z_0, z_1, ...z_{t-1}) \in Y_t$, $x, x' \in X_t$, then $x \hat{\succsim}_{y_t} x'$ iff $(z_0, z_1, ...z_{t-1}, x) \succsim_{y_t}$ $(z_0, z_1, ..., z_{t-1}, x')$.

Remark 3.5.9. *An individual with naive quasi-hyperbolic preference violates the temporal consistency axiom. A sophisticated one does not.*

Lemma 3.5.10 (Lemma 4 in KP)**.** *Axioms 2.1, 2.2, 2.3, and 3.1 are necessary and sufficient for there to* *exist functions* U_{y_t} *as in Lemma 3.5.6 and, for fixed* $\{U_{y_t}\}$*, unique functions*

$$W_{y_t}\colon \left\{(z,\gamma)\in Z_t\times\mathbb{R} : \gamma=\hat{U}_{(y_t,z_t)}(x) \text{ for some } x\in X_{t+1}\right\}\to\mathbb{R}$$

which are strictly increasing in their second argument and which satisfy

$$U_y(z,x)=W_y(z,\hat{U}_{(y,z)}(x))=\max_{d\in x}W_y(z,E_d[U_{(y,z)}]) \tag{3.4}$$

that is, W *is a non-linear intertemporal aggregator.*

Proof. **Sufficiency:** Check that x,x' such that $\hat{U}_{(y,z)}(z)=\hat{U}_{(y,z)}(x')$ is equivalent by definition to $x\,\hat{\sim}_{(y,z)}\,x'$, that by Axiom 3.1 gives $(z,x)\sim_y(z,x')$, which is by definition $U_y(z,x)=U_y(z,x')$. The same chain can be made with strict binary relations.

Necessity: Need just to verify that Axiom 3.1 holds. If for $x,x'\in X_{t+1}$, $x\,\hat{\succsim}_{(y,z)}\,x'$, then $\hat{U}_{(y,z)}(x)\geq\hat{U}_{(y,z)}(x')$, and, by the monotonicity of W_y, $W_y(z,\hat{U}_{(y,z)}(x))\geq W_y(z,\hat{U}_{(y,z)}(x'))$, which can be rewritten as $U_y(z,x)\geq U_y(z,x')$, or $(z,x)\succsim_y(z,x')$. Proceed with the same argument with strict relation $\succ$ to show the other direction. We will use *strict* monotonicity of W there. □

Theorem 3.5.11 (Theorem 1 in KP)**.** *Axioms 2.1, 2.2, 2.3, and 3.1 are necessary and sufficient for there to exist a continuous function* $U\colon Y_{T+1}\to\mathbb{R}$ *and, for* $t=0,...T-1$*, continuous functions* $W_t\colon Y_t\times Z_t\times\mathbb{R}\to\mathbb{R}$*, strictly increasing in their third argument, so that if we define* $U_{y_T}(z_T):=U(y_T,z_T)$ *and, recursively,*

$$U_{y_t}(z_t,x_{t+1}):=\max_{d\in x_{t+1}}W_t(y_t,z_t,E_d[U_{(y_t,z_t)}]),$$

then for all y_t *and* $d,d'\in D_t$*,* $d\succsim_{y_t}d'$ *iff* $E_d[U_{y_t}]\geq E_{d'}[U_{y_t}]$*.*

Proof. Let U_{y_0} be as guaranteed by Lemma 3.5.6. It is continuous. For each y_t, there exist $x',x''\in X_t$ such that $x'\,\hat{\succsim}_{y_t}\,x\,\hat{\succsim}_{y_t}\,x''$ for all $x\in X_t$. Fix the version of U_{y_t} as in Lemma 3.5.6 so that $\hat{U}_{y_t}(x')=U_{y_0}(y_t,x')$ and $\hat{U}_{y_t}(x'')=U_{y_0}(y_t,x'')$.

... □

3.6 Epstein, Zin (1989)

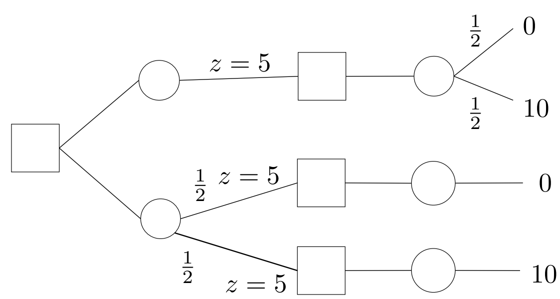
This paper builds on Kreps-Porteus in two aspects:

1. infinite horizon;

2. changed expected utility to betweenness and RDU.

Outside of the DT profession, the contribution the authors are best known for is Epstein-Zin preferences. EZ preferences are the special case of a Kreps-Porteus representation with $U(z) = z^\alpha$ and $W(y, z, \gamma) = W(z, \gamma) = z^\sigma + \delta \left(\gamma^{\frac{1}{\alpha}}\right)^\sigma$.

Example 3.6.1. $Z = \{0, 10\}$.



If $\alpha = \frac{1}{2}$, $\sigma = \frac{1}{2}$, then I am indifferent between tossing the coin earlier and later. If $\alpha = \frac{1}{2}$, $\sigma = 1$, I prefer to toss earlier (lower branch). If $\alpha = 1$, $\sigma = \frac{1}{2}$, I prefer to toss later (upper branch).

Lecture 4

Subjective Probability

Thus far, all models have assumed knowledge of objective probabilities over events. This section responds to the criticism that such objective probabilities may not always exist, and develops the machinery to handle situations of subjective probability. More precisely, models of the following class are concerned with two questions:

1. Under what conditions are an individual's beliefs over the relative likelihood of events consistent with classical probability theory? That is, when can a binary relation of likelihood $\succeq^*$ over events be represented by a classical probability measure μ such that $E \succeq^* F$ if and only if $\mu(E) \geq \mu(F)$?

2. When can one derive probabilities from choice? That is, under what conditions are choices over subjectively uncertain acts consistent with probabilistically sophisticated beliefs over event likelihoods?

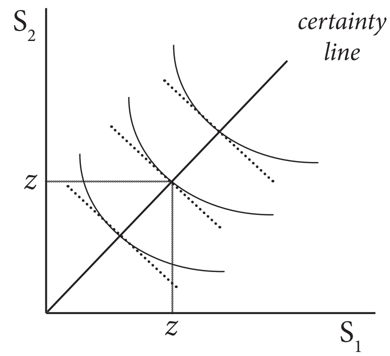
The primitives are a state space S, outcomes Z, *acts* defined as $\mathcal{F} = Z^S$, and a preference relation $\succ$ over $\mathcal{F}$. Note the innovation here is preference over acts instead of preference over outcomes. Typical elements are $E, F \subseteq S$, $x, y \in Z$, and $f, g, h \in \mathcal{F}$. We assume states to be mutually exclusive.

Definition *Subjective Expected Utility* holds on a set $\mathcal{F}^S$ if there exist nonnegative probabilities $\{p_s\}_{s \in S}$ summing to 1 and a utility function $V : \mathcal{F} \to \mathbb{R}$ such that,

$$V(f) = \sum_{s \in S} u(f(s)) \cdot p(s).$$

Note that this is a special case of additive utility in which probabilities are independent of states.

Definition *Additive Utility* (also known as *State-Dependent SEU*) holds on a set $\mathcal{F}^S$ if there exists a utility



function $V : \mathcal{F} \to \mathbb{R}$ such that,

$$V(f) = \sum_{s \in S} u_s(f(s)).$$

4.1 Werner, 2005

Werner (2005) develops an axiomatic foundation for risk-averse expected utility. The main result is that if (1) preferences exhibit risk aversion with respect to some probabilities and (2) they satisfy the Independence Axiom, then there exists an EU representation with respect to these probabilities. Let $Z = \mathbb{R}_+$ and S be finite.

Definition $\succ$ is *risk averse* if and only if there exist $p \in \Delta(S)$ such that for all f, $E_p f \succeq f$.

Under monotonicity, this is equivalent to: $\succ$ is risk averse if and only if there exist $p \in \Delta(S)$ for all z such that $f \succeq z$ implies $E_p f \succeq z$.

Theorem 4.1.1 (Werner)**.** *Suppose that* $|S| \geq 3$*. Then,*

$$\succ \text{ admits EU representation with concave, strictly increasing } u \quad \longleftrightarrow \quad \begin{cases} \text{(1) Weak Order} \\ \text{(2) Strict Monotonicity} \\ \text{(3) Continuity} \\ \text{(4) P2} \\ \text{(5) Risk Aversion} \end{cases}$$

(Refer to the Savage section for elaboration on P2, which says that $f_E g \succ h_E g$ if and only if $f_E g' \succ h_E g'$.) The proof for this theorem is very roughly as follows. Suppose that U satisfies the Independence Axiom. It follows that U has state-separable representation $U_i = \sum_{s=1}^{S} u_i(s)$. Since agents are risk averse, optimization requires $u_1'(z) = cu_2'(z)$ for all z. If we know that we can integrate, then,

$$\begin{aligned}
u_2(z) &= c_2 u_1(z) + k_2 \\
u_3(x) &= c_3 u_1(z) + k_3 \\
&\vdots \\
u_s(x) &= c_s u_1(z) + k_s
\end{aligned}$$

where $c_s = \frac{p_2}{p_1}$ for all s. It follows that $\sum u_s(f_s) = \sum p_s u(f_s) + k$, where $k = \sum_{s=1}^{S} k_s$, and we have the desired EU representation. This is illustrated roughly in figure 4.1. The axes represent consumption in either of two states S_1 or S_2; curved lines represent indifference curves. The slope at tangency is,

$$-\frac{u_1(z)}{u_2(z)} = -\frac{u'(z)p_1}{u'(z)p_2} = -\frac{p_1}{p_2},$$

which is constant along the certainty line.

4.2 Kobberling and Wakker, 2003

This paper presents an axiomatization for expected utility based on a tradeoff technique.

Definition If for some $x, y, w, z \in Z$, $f, g, \in \mathcal{F}$, and $E \subseteq S$ non-null, it holds that $x_E f \sim y_E g$, $z_E f \sim w_E g$, then write,

$$x \ominus y \overset{t}{\sim} z \ominus w,$$

where $\overset{t}{\sim}$ denotes tradeoff indifference (note that formally this depends on f, g, E).

The interpretation is that receiving x instead of y is the same to the agent as receiving z instead of w – in both cases the difference exactly offsets the difference between f and g. We can think of $\overset{t}{\sim}$ therefore as capturing an aspect of the strength of the agent's preferences. Under state independence,

$$u(x)p(E) + \sum_{s \in E^c} u(f(s))p(s) = u(y)p(E) + \sum_{s \in E^c} u(g(s))p(s)$$
$$u(z)p(E) + \sum_{s \in E^c} u(f(s))p(s) = u(w)p(E) + \sum_{s \in E^c} u(g(s))p(s)$$

$$
\begin{array}{ccc}
x_1 \ominus x_0 & \overset{t}{\sim} & z \ominus y \\
\overset{t}{\wr} & & \\
x_2 \ominus x_1 & & \\
\overset{t}{\wr} & & \\
x_3 \ominus x_2 & & \\
\overset{t}{\wr} & & \\
\cdots & &
\end{array}
$$

Figure 4.1: The outcomes $x_0, x_1, ...$ form a *standard sequence.*

which yields the natural $u(x) - u(y) = u(z) - u(w)$. This idea is generalized below:

Axiom 4.2.1 (Tradeoff Consistency)**.** $x \ominus y \overset{t}{\sim} z \ominus w$ *implies for all* $x' \succ x$, $y' \succ y$, $z' \succ z$, $w' \succ w$,

- $x' \ominus y \overset{t}{\nsim} z \ominus w$
- $x \ominus y' \overset{t}{\nsim} z \ominus w$
- *...and so forth.*

Essentially, this axiom says that improving an outcome in any $\overset{t}{\sim}$ *relationship breaks that relationship. Additionally, this implies that* $\overset{t}{\sim}$ *does not depend on* E *(but still depends on* f*,*g*).*

A utility function can then be constructed using a "sawtooth method". The idea is first to create a *standard sequence* of prizes $x_0, x_1, ...$ that are equally spaced in utility units, as represented in figure 4.2. The "gauge outcomes" $z \succ y$ serve to make sure the standard sequence does not consist of equivalent outcomes. This is illustrated in figure 4.2. We can then assign utilities $u(x_0) = 0$, $u(x_1) = 1$, $u(x_2) = 2$, and so forth, as in figure 4.2.

Theorem 4.2.2 (Wakker)**.** *Suppose that* Z *is a connected and separable topological space. Then,*

$$
\succ \text{ admits SEU-representation with continuous } u \longleftrightarrow \begin{cases} \text{(1) Weak Order} \\ \text{(2) Continuous} \\ \text{(3) Tradeoff Consistency} \end{cases}
$$

Note that the approaches thus far have been to begin with derivation of utility, and subsequently to construct probabilities. This procedure will be reversed following as we discuss Savage.

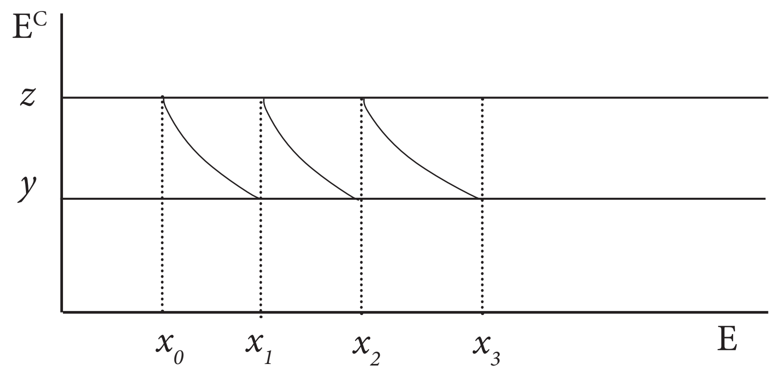


Figure 4.2: The standard sequence satisfies $x_{k+1} \ominus x_k \overset{t}{\sim} z \ominus y$ for all $k, k+1$.

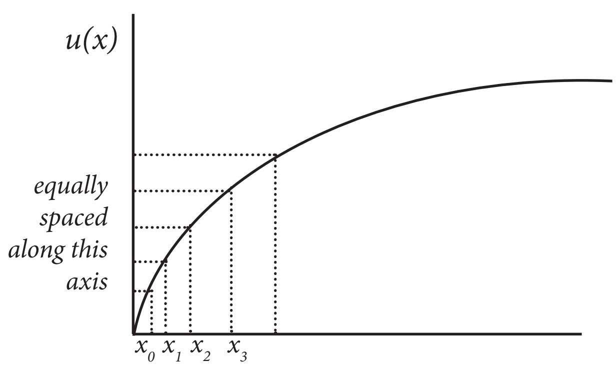


Figure 4.3: The standard sequence is equally spaced in utility units.

4.3 Savage, 1959

4.3.1 Preliminaries

Savage (1959) is a characterization of *probabilistically sophisticated expected utility maximizers.* That is, Savage takes purely subjective acts (this differentiates Savage's work from Anscombe-Aumann's, which will be discussed shortly), and derives utility representation of the form,

$$f \succ g \longleftrightarrow E_\mu[u(f(s))] > E_\mu[u(g(s))]$$

where f and g are acts, $u(\cdot)$ is utility of EU form, and μ is a classical probability measure over states. The Savage Axioms are as follows:

(P1) *Weak Order.* $\succ$ on $\mathcal{F}^S$ is a weak order.

(P2) *Sure Thing Principle.* $f_E g \succ h_E g$ if and only if $f_E g' \succ h_E g'$.

In words: if two acts imply different subacts over an event E, but the same subact over the complement of E, the ranking of these acts will not depend on the common subact. Note the close relation to Expected Utility – an immediate implication is that preferences are separable across mutually exclusive events.

(P3) *Event Monotonicity.* For E non-null, $x, y \in Z$, and $h \in \mathcal{F}^S$,

$$x_E h \succ y_E h \text{ if and only if } x \succ y.$$

In words: replacing an outcome over a non-null event by a more preferred outcome yields a strictly better outcome.

(P4) *Weak Comparative Probability.* If $x, x', y, y' \in Z$ and $x \succ y$, $x' \succ y'$, then

$$x_E y \succ x_F y \to x'_E y \succ x'_F y'.$$

In words: when "revealing" probability estimations, an agent's likelihood rankings are independent of the specific prizes involved.

(P5) *Non-triviality.* There exist $f, g \in \mathcal{F}^S$ such that $f \succ g$.

In words: the relation $\succeq$ is not trivial; there exists some act that is strictly preferred to another.

(P6) *Small-event continuity.* For all $f, g, h \in \mathcal{F}$ with $f \succ g$, there exists a partition of S denoted $(E_1, ..., E_n)$ such that $h_{E_i} f \succ g$ and $f \succ h_{E_i} g$ for all $i = 1, ..., n$.

In words: The set S can be partitioned into small enough events such that altering either act on just one of these events to the act h is not enough to reverse the original ranking.

Exercise 4.3.1. *If you are so inclined:*

1. *Show that* $P1 + P2 + P5 + P6$ *imply that* S *is infinite.*
2. *Show that* $P6$ *implies that for all* $s \in S$, $\{s\}$ *is null.*

Definition $\mu : 2^S \to [0, 1]$ is *finitely additive* if for all $E, F \subseteq S$, $E \cap F = \emptyset$,

$$\mu(E \cup F) = \mu(E) + \mu(F).$$

Definition E is an *atom* if and only if $\mu(E) > 0$ implies that for all $F \subseteq E$, either $\mu(F) = 0$ or $\mu(F) = \mu(E)$.

In words: we cannot split E into two events of positive probability.

Definition μ is *nonatomic* if it has no atoms.

Definition μ is *convex-ranged* if for all $E \subseteq S$ such that $\mu(E) > 0$ and for all $\lambda \in [0, 1]$, there exists $F \subseteq E$ such that $\mu(F) = \lambda \mu(E)$.

4.3.2 Representation

Theorem 4.3.2 (Savage)**.** $\succ$ *satisfies Axioms (P1) - (P6) on* $\mathcal{F}^S$ *if and only if there exist a function* $u : Z \to \mathbb{R}$ *and a finitely additive and convex-ranged* $\mu \in \Delta(S)$ *such that,*

$$f \succ g \text{ if and only if } \int u \circ f \, d\mu > \int u \circ g \, d\mu.$$

In this case, μ *is unique, and* u *is unique up to positive affine transformations.*

A few remarks:

1. If μ is σ-additive on some σ-algebra Σ, then μ nonatomic $\longleftrightarrow$ μ convex-ranged.
2. $\mathcal{F}^S = \{f : S \to Z, \text{ simple}, \Sigma - \text{measurable}\}$ then everything goes but with another method of proof. See Kopylov JET '07: Σ-algebra, λ-system.

3. Continuity hypothesis $\rightarrow \Sigma = 2^S \rightarrow \mu$ is not σ-additive.

Axiom 4.3.3 (Monotone Continuity). *For all $f \succ g$, $x \in Z$, and $\{E_n\} \subset S$ such that $E_{n+1} \subseteq E_n$ $\forall$ n and $\cap_n E_n = \emptyset$, there exists a N such that for all $n > N$,*

$$x_{E_n} f \succ g \quad \text{and} \quad f \succ x_{E_h} g.$$

That this axiom is close to (P6) but not equivalent. To get (P6) we need another axiom:

Axiom 4.3.4 (Nonatomicity). *If E is non-null, there exist non-null events $A, B \subset S$ such that $A \cup B = E$ and $A \cap B = \emptyset$.*

Theorem 4.3.5. *P1-P5 + MC + NA $\longrightarrow$ EU.*

4.3.3 Proof

We will now walk through the proof of Savage's representation theorem. Recall that the goal is to go from preferences $\succ$ over acts $\mathcal{F}^S$ satisfying (P1)-(P6) to a probability measure μ and Expected Utility function u such that,

$$f \succ g \longleftrightarrow E[u(f(s)), \mu] > E[u(g(s)), \mu]$$

The strategy is as follows:

1. Establish the conditions under which subjective beliefs yield a unique probability measure.
2. Show that if preference over acts satisfies (P1)-(P6), then the conditions in (1) are fulfilled, and we can construct such a probability measure μ.
3. Define from μ a preference relation $\widehat{\succ}$ over distributions $\Delta^S(Z)$ and show that the axioms of Expected Utility are satisfied.
4. Invoke von Neumann-Morgenstern.

Note that these steps exactly correspond to the dashed path in Figure 4.4. That is; Subjective Expected Utility (which maps directly from acts to values) is equivalent to construction of probability (which maps from acts to distributions) coupled with a standard EU function (which maps from distributions to values).

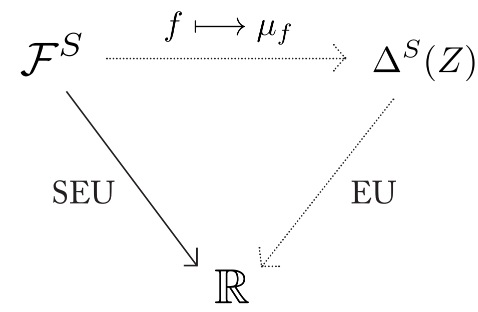


Figure 4.4: SEU is equivalent to composition of a probability measure μ_f and utility u of EU form.

As noted earlier, this is slightly different from the method of Werner and Wakker, in which utility is constructed before probability.

STEP 1: Let $\succ^*$ represent the agent's subjective interpretation of likelihood. That is, $A \succ^* B$ if and only if $x_A y \succ x_B y$ for some $x \succ y$. We are interested in obtaining a finitely additive measure μ such that $A \succ^* B$ if and only if $\mu(A) > \mu(B)$. μ would then be a numerical representation of the relation on the subsets. The necessary and sufficient conditions for such representation are the following axioms:

(F1) For all A nonempty, $A \succ^* \emptyset$.

(F2) $S \succ^* \emptyset$.

(F3) $\succ^*$ is a weak order.

(F4) If $A \cap C = B \cap C = \emptyset$, then $A \succ^* B \longleftrightarrow A \cup C \succ^* B \cup C$.

(F5) $A \succ^* B \rightarrow$ there exists a partition of the state space $C_1, ..., C_n$ such that for all i, $A \succ^* B \cup C_i$.

Theorem 4.3.6. *$\succ^*$ satisfies Axioms (F1) - (F5) if and only if there exists a unique convex-ranged and finitely additive measure μ on 2^S that represents $\succ^*$.*

STEP 2. Section 14.3 of Fishburn shows that fulfillment of (P1) - (P6) implies fulfillment of (F1) - (F5). Hence there exists for each $\succ$ a unique measure $\mu : \mathcal{F}^S \rightarrow \Delta^S(Z)$ as in the top line of the triangle in figure 4.4.

STEP 3. Now we will construct from μ a preference relation $\widehat{\succ}$ over $\Delta^S(Z)$.

Definition If $f \in \mathcal{F}^S$, then let $\mu_f \in \Delta^S(Z)$ be defined by,

$$\mu_f(z) = \mu(\{s \in S | f(s) = z\}).$$

That is, $\mu_f = \mu \circ f^{-1}$.

Definition For $P, Q \in \Delta^S(Z)$, define $\widehat{\succ}$ on $\Delta^S(Z)$ by,

$$P \,\widehat{\succ}\, Q \quad \text{if and only if} \quad f \succ g \text{ whenever } P = \mu_f, Q = \mu_g.$$

We need to show that $\hat{\succ}$ satisfies the von Neumann-Morgenstern axioms.

Theorem 4.3.7. *Axioms (P1) - (P6) imply* $(\mu_f = \mu_g \rightarrow f \sim g)$.

Combined with (P1), this implies that $\widehat{\succ}$ is a weak order.

Theorem 4.3.8. $(P, W, R \in \Delta^S(Z), \alpha \in (0, 1),$ *P1-P6) imply,*

$$P \,\widehat{\succ}\, W \quad \longleftrightarrow \quad \alpha P + (1 - \alpha)R \,\widehat{\succ}\, \alpha Q + (1 - \alpha)R).$$

Thus, $\widehat{\succ}$ satisfies the Independence Axiom.

Theorem 4.3.9. $(P, Q, \in \Delta^S(Z), f \in \mathcal{F}^S, P \,\widehat{\succ}\, Q, P \,\widehat{\succeq}\, \mu_f \hat{\succeq} Q)$ *imply there exists a unique* $\alpha \in [0, 1]$ *such that,*

$$\mu_f \,\widehat{\succ}\, \alpha P + (1 - \alpha)Q.$$

Thus, $\widehat{\succ}$ satisfies the Archimedean Axiom.

STEP 4: Invoke the von Neumann-Morgenstern representation theorem. This yields the desired EU function u, unique as usual up to an affine transformation.

4.4 Machina and Schmeidler, 1992

Machina and Schmeidler (1992) extend Savages' work to probabilistically sophisticated agents with *non-Expected Utility*. That is, in figure 4.4 we generalize the mapping from $\Delta^S(Z)$ to $\mathbb{R}$ to allow for utility of non-EU form. See figure 4.5 for a comparison.

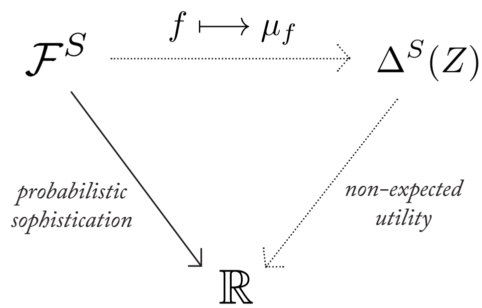


Figure 4.5: SEU is equivalent to generation of a probability measure μ_f followed with u of EU form.

(P4*) *Strong Comparative Probability* For all events $A, B \subset S$, $A \cap B = \emptyset$, and outcomes $x, y, x', y' \in Z$ satisfying $x \succ y$, $x' \succ y'$,

$$x_A y_B g \succeq y_A x_B g \longrightarrow x'_A, y'_B h \succeq y'_A x'_B h.$$

Theorem 4.4.1. *$\succ$ satisfies Axioms P1, P3, P4*, P5, P6 if and only if there exists a unique finitely additive and convex-ranged $\mu \in \Delta(S)$ such that,*

$$\mu_f = \mu_g \longrightarrow f \sim g.$$

and a nonconstant mapping $M : \Delta^S(Z) \to \mathbb{R}$ satisfying FOSD such that,

$$f \succ g \longleftrightarrow M(\mu_f) > M(\mu_g).$$

4.5 Summary

	Expected Utility	Objective Probabilities	Subjective Probabilities
von Neumann-Morgenstern	Yes	Yes	No
Anscombe-Aumann	Yes	Yes	Yes
Savage	Yes	No	Yes
Machina-Schmeidler	No	No	Yes

Lecture 5

Ambiguity Aversion

5.1 Three Color Paradox

Example 5.1.1. *Suppose there is an urn with* 90 *colored balls of which* 30 *are known to be red, and the remaining sixty are known to be either green or yellow. Thus, the sample space for bets on the color of a ball chosen from the urn is* $S = \{R, G, Y\}$.

Let f *be a bet that yields a dollar if a red ball is chosen and* g *be a bet that yields a dollar if a green ball is chosen. Let* f' *be a bet that yields a dollar if a red or yellow ball is chosen, and let* g' *be a bet that yields a dollar if a green or yellow ball is chosen. Then preferences of the form* $f \succ g$ *and* $f' \prec g'$ *violate* $P2$ *from the Savage setting.*

	R	G	Y
f	1	0	0
g	0	1	0
f'	1	0	1
g'	0	1	1

To see this, define h *as a bet that always yields nothing and* h' *as a bet that always yields a dollar. Then* $f = f_{\{Y\}}h$[1], $g = g_{\{Y\}}h$ *and* $f' = f_{\{Y\}}h'$, $g' = g_{\{Y\}}h'$, *however the axiom requires that* $f_E h \succ g_E h \iff f_E h' \succ g_E h'$.

[1]That is f is unchanged by replacing its outcome on the set $\{Y\}$ with that of h on this set.

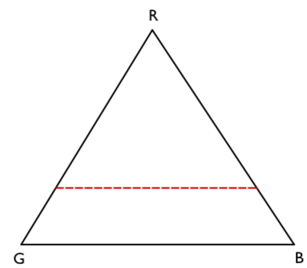


Figure 5.1: A probability with one known component is a line parallel to one side of the unit simplex

5.2 Maxmin Expected Utility (MEU)

One resolution to the example above is that the agent has a set of probabilities C, and evaluates each act with its worst case probability. Consider a utility function over acts f

$$V(f) = \min_{p \in C} \sum u(f(s))p(s) \tag{5.1}$$

where $C \subseteq \Delta(S)$ is a non-empty, closed and convex set, and maximization is over acts f.

Example 5.2.1. *Consider* $S = \{R, G, B\}$, $u \equiv \mathbf{1}_{\{G\}}$ *and*

$$C = \left\{ p \,|\, p(R) = \frac{1}{3} \right\}$$

This set is graphically represented in Figure 5.1.

5.2.1 Support functions

The unit simplex is also useful for representing sets C. In the VNM setting, as shown in Figure 5.2, indifference curves are parallel lines.

However, an act can also be represented as a line on the simplex, holding the utility function fixed, where the probability being used to assess it changes, as the value of an act is linear in probabilities. With each act, one can find the member of a set C that minimizes its value. These points are captured by the support function. As illustrated in Figure 5.3 this process makes C indistinguishable from its convex hull, and thus in order to have any sort of uniqueness C is typically required to be convex, closed and non-empty.

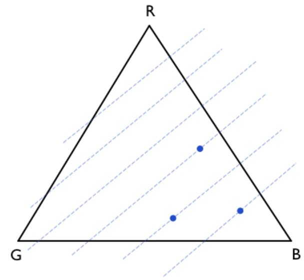


Figure 5.2: In the VNM setting a utility function is represented by parallel indifference curves

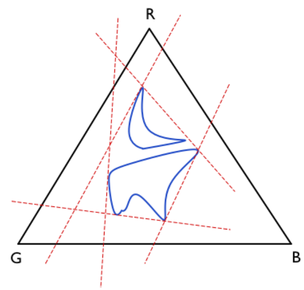


Figure 5.3: Finding points that are minimizers for some acts, C is indistinguishable from its convex hull

Definition For $C \subseteq \Delta(S)$ a non-empty, closed and convex set define the *support function*:

$$I(\xi) = \min_{p \in C} \int_S \xi dp \tag{5.2}$$

Note that $I : \mathbb{R}^S \mapsto \mathbb{R}$.

5.2.2 Properties of support functions

Which I's are support functions? How can the set C be recovered given its support function I? We prove a subset of the list of properties below and leave the rest as exercises:

1. **Concavity**

Proof:

$$\begin{aligned} I\left(\alpha\xi + (1-\alpha)\zeta\right) &= \min_{p\in C}\int_S \left(\alpha\xi + (1-\alpha)\zeta\right)dp \\ &= \int_S \left(\alpha\xi + (1-\alpha)\zeta\right)dp^* \\ &= \alpha\int_S \xi dp^* + (1-\alpha)\int_S \zeta dp^* \\ &\geq \alpha\min_{p\in C}\int_S \xi dp + (1-\alpha)\min_{p\in C}\int_S \zeta dp = \alpha I(\xi) + (1-\alpha)I(\zeta) \quad \square \end{aligned}$$

2. **Positively homogeneous (scale invariant)**

$$I(\alpha\xi) = \alpha I(\xi)\,,\; \alpha > 0$$

3. **Translation invariant**

$$I(\xi + k) = I(\xi) + k\,,\; k \in \mathbb{R}$$

4. **Normalization**

$$I(k) = k$$

5. **Monotonicity**

$$\xi(s) \geq \zeta(s)\,\forall s \in S \Rightarrow I(\xi) \geq I(s)$$

6. **Continuity**

Proof: Consider a converging sequence $\xi^n \to \xi$ and minimizers p^n and p^*.

$$\begin{aligned} I(\xi) &= \int_S \xi dp^* \leq \int_S \xi dp^n \\ I(\xi^n) &= \min_{p\in C}\int_S \xi^n dp = \int_S \xi^n dp^n \leq \int_S \xi^n dp^* \\ \xi^n \to \xi &\Rightarrow \int_S \xi^n dp^* \to \int_S \xi dp^*\,,\; \int_S \xi^n dp^n \to \int_S \xi dp^n \quad \square \end{aligned}$$

5.2.3 Related theorems

A math theorem based on these properties:

Theorem 5.2.2 (Rockafeller Cor 13.2.2)**.** *Properties* $(1),(2) \iff \exists C \in \mathbb{R}^n$ *convex and non empty s.t.*

$$I(\xi) = \inf_{\mu \in C} \int \xi d\mu$$

A decision theory theorem using these properties:

Theorem 5.2.3 (Efe Ok Section H1.3)**.** *Properties* $(1),(2),(3),(4),(5) \iff \exists C \in \Delta(S)$ *convex, closed and non empty s.t.*

$$I(\xi) = \min_{p \in C} \int \xi dp$$

5.3 Choquet Expected Utility

Choqeut Expected Utility uses the concept of a capacity (which also appears in cooperative game theory) to capture the degree of confidence in an event in a fuzzier manner than a probability.

Definition $\nu : 2^S \mapsto [0,1]$ OR $\nu : \Sigma \mapsto [0,1]$ is a *capacity* if:

1. $\nu(\emptyset) = 0$, $\nu(S) = 1$
2. ν is monotone

$$E \supseteq F \Rightarrow \nu(E) \geq \nu(F)$$

Example 5.3.1. *Suppose that one is presented a container with colored balls inside, with the information that out of* 90 *balls,* 30 *are red, and the remaining* 60 *are either green or yellow. This can be represented by* $S = \{R, G, Y\}$ *with the following capacities*

$$\nu(\{R\}) = \frac{1}{3}\,,\, \nu(\{R,G\}) = \nu(\{R,Y\}) = \frac{1}{3}\,,\, \nu(\{G\}) = \nu(\{Y\}) = 0\,,\, \nu(\{G,Y\}) = \frac{2}{3}.$$

Definition Suppose the outcomes of a simple act ξ are ordered $x_1 \geq x_2 \geq \ldots \geq x_n$ with $x_{n+1} = 0$, and define the event that x_i occurs as E_i. Then the *Choquet Integral* with respect to a capacity is

$$\int \xi d\nu = \sum_{i=1}^{n} (x_i - x_{i+1}) \nu \left(\bigcup_{j=1}^{i} E_j \right). \tag{5.3}$$

This construction is like the Lebesgue integral, as illustrated in Figure 5.4.

Definition A capacity ν is *convex* (supermodular) if and only if

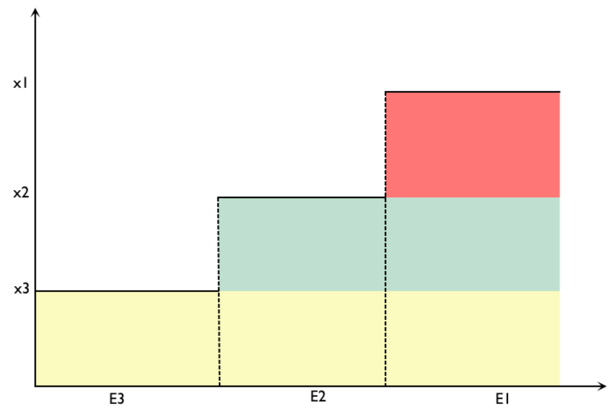


Figure 5.4: Constructing the Choquet integral: the difference $x_2 - x_3$ is weighted by $\nu(E_2 \cup E_1)$

$$\nu(E \cup F) + \nu(E \cap F) \geq \nu(E) + \nu(F).$$

Convex capacities will correspond to ambiguity aversion.

Theorem 5.3.2 (Schmeidler '89)**.** *Let $I(\xi) = \int \xi d\nu$. Then the following statements are equivalent:*

1. *I is concave*
2. *ν is convex*
3. *I can be expressed as*

$$I(\xi) = \min_{p \in C} \int \xi dp \tag{5.4}$$

where C is the core of ν:

$$\{p \in \Delta S | p(E) \geq \nu(E) \, \forall E\}.$$

This illustrates that a convex capacity can be expressed both in the Choquet set up, and in the MEU set up through the core of the capacity. What kinds of sets C can arise from Choquet integrals? As illustrated in Figure 5.5, in the three object case there are six constraints implied by being in the core of a capacity, and the resulting set C can only be a polygon. Thus, if C is a circle, for example, then I defined using this set is not a Choquet integral.

Note that it is possible to create a capacity by distorting a measure. Begin with $p \in \Delta(S)$ and take $\psi : [0,1] \mapsto [0,1]$ and define the capacity by

$$\nu(E) = \psi(p(E)).$$

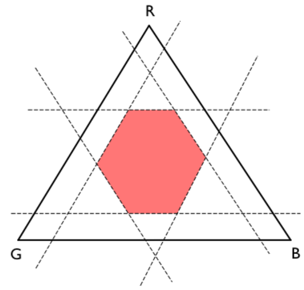


Figure 5.5: What the core of a capacity can look like

Then concavity of ψ implies convexity of ν.

5.4 Comonotic acts

Definition Two acts ξ, ζ are *comonotonic* if and only if $\neg\exists s, s' \in S$ s.t.

$$\xi(s) > \xi(s') \wedge \zeta(s) < \zeta(s').$$

Definition A support function I satisfies *comonotonic additivity* if and only if for any comonotonic acts ξ, ζ

$$I(\xi + \zeta) = I(\xi) + I(\zeta).$$

Theorem 5.4.1 (Schmeidler 1986)**.** *I is normalized, monotone and satisfies comonotinic additivity if and only if there exists ν a capacity with*

$$I(\xi) = \int \xi d\nu. \tag{5.5}$$

5.5 Anscombe-Aumann and modifications

5.5.1 The Anscombe-Aumann set up and axioms

Consider S, Z finite and consider $\mathcal{F} = (\Delta Z)^S$, with preferences $\succ$ on $\mathcal{F}$. One way to think about this set up is as a combination of horse race lotteries (with subjective probability) and roulette lotteries (with objective probabilities), where the subjective probability is derived by making comparisons between these two types of lotteries.

Axiom 5.5.1. *$\succ$ is a weak order.*

Axiom 5.5.2 (Independence). *For all $f, g, h \in \mathcal{F}$, $\alpha \in (0,1)$, $f \succ g$ if and only if,*

$$\alpha f + (1-\alpha)h \succ \alpha g + (1-\alpha)h$$

where addition is performed by state.

Axiom 5.5.3 (Mixture continuity). *For all $f, g, h \in \mathcal{F}$ the sets*

$$\{\alpha \in [0,1] | \alpha f + (1-\alpha)g \succeq h\}$$

$$\{\alpha \in [0,1] | \alpha f + (1-\alpha)g \preceq h\}$$

are closed.

Axiom 5.5.4 (Monotonicity). *For all $f, g \in \mathcal{F}$*

$$f(s) \succeq g(s) \forall s \implies f \succeq g$$

Axiom 5.5.5 (Uncertainty aversion).

$$f \sim g \implies \alpha f + (1-\alpha)g \succeq f$$

Axiom 5.5.6. *For some $f, g \in \mathcal{F}$, $f \succ g$.*

To motivate the uncertainty aversion axiom, consider the following example:

Example 5.5.7. *Suppose that there are two urns, urn I with a hundred balls that are either black or white, and urn II with fifty black and fifty white balls. The sample space is therefore $S = \{B, W\}$. Define the constant δ_x to mean getting \$x with certainty, and B_{II} to mean, for example, choosing a black ball from the second urn. Suppose all bets yield ten dollars if the appropriate outcome occurs. Then we have that B_{II} is a random variable with equal weights on the two outcomes δ_0 and δ_{10}:*

$$B_{II} \sim \frac{1}{2}\delta_0 + \frac{1}{2}\delta_{10} \sim W_{II}$$

and similarly

$$B_I \sim \delta_{10\,\{B\}}\delta_0 \sim W_I \sim \delta_{10\,\{W\}}\delta_0 \prec B_{II}.$$

We have that a mixture of two bets on the first urn that are indifferent to each other can be better than either for an ambiguity averse agent as some ambiguity is hedged away:

$$\frac{1}{2}B_I + \frac{1}{2}W_I = B_{II}.$$

5.5.2 The Anscombe-Aumann theorem

Theorem 5.5.8. *$\succ$ satisfies axioms 5.5.1, 5.5.2, 5.5.3, 5.5.4, 5.5.6 if and only if $\exists u : \Delta(Z) \mapsto \mathbb{R}$, $p \in \Delta(S)$ with u affine s.t. $f \succ g$ iff*

$$\sum_s u(f(s))p(s) > \sum_s u(g(s))p(s) \tag{5.6}$$

p is unique, and u is unique up to positive affine transformations.

The proof of this theorem proceeds in 5 steps:

1. $\succ$ on $\Delta(Z)$ is represented by $u : \Delta(Z) \mapsto \mathbb{R}$ affine. Denote the range of the function as $U := u(\Delta(Z))$.

2. Define $\succ^*$ on U^S: $\xi \succ^* \zeta$ iff for some $f \in \mathcal{F}$ s.t. $u \circ f = \xi$, $g \in \mathcal{F}$ s.t. $u \circ g = \zeta$ $f \succ g$. By monotonicity $\succ^*$ is a weak order.

3. For any $\xi \in U^S$, define

$$\bar{\xi} = \max_{s \in S} \xi(s)$$

and $\underline{\xi}$ as the correspondingly defined minimum. Define

$$A = \left\{\alpha \in [0,1] | \alpha\xi + (1-\alpha)\bar{\xi} \succeq \xi\right\}$$
$$B = \left\{\alpha \in [0,1] | \alpha\xi + (1-\alpha)\underline{\xi} \preceq \xi\right\}$$

Note that A and B are not empty because $\bar{\xi} \succeq^* \xi \succeq^* \underline{\xi}$ by monotonicity of $\succ^*$. Moreover A and B are closed by mixture continuity of $\succ^*$. Now as $A \cup B = [0,1]$ which is a connected set, $A \cap B \neq \emptyset$. Define $I(\xi)$ to be the indifferent element of U according to $\succ^*$.

If I is linear, then we have

$$I(\xi) = \sum_s \xi(s)p(s)$$

and if I is monotone, $p(s) \geq 0 \; \forall s$. Finally if I is normalized (i.e. $I(k) = k$) then $\sum p(s) = 1$. We will show linearity, which follows from positive homogeneity and additivity.

4. I is positive homogenous: WLOG assume that $[-1, 1] \subseteq U$. If $I(\xi) = k$, then we need to show that $I(\alpha\xi) = \alpha k$. Suppose that, instead

$$I(\alpha\xi) > \alpha k \iff I(\alpha\xi + (1-\alpha)0) \geq \alpha k + (1-\alpha)0 = I(\alpha k + (1-\alpha)0)$$

Thus $\alpha\xi + (1-\alpha)0 \succeq^* \alpha k + (1-\alpha)0$. Let $0_* \in \Delta(Z)$ be s.t. $u(0_*) = 0$. Let f be such that $u \circ f = \xi$, and π s.t. $u(\pi) = k$. Using that

$$\alpha f + (1-\alpha)0_* \succeq \alpha\pi + (1-\alpha)0_*$$

(by independence) $\iff f \succeq \pi \iff \xi \succeq^* k \iff I(\xi) \geq k$.

5. I is additive: $I(\xi + \zeta) = I(\xi) + I(\zeta)$. Begin by extending I to $\mathbb{R}^S$ using positive homogeneity, and re-write as

$$I\left(\frac{1}{2}2\xi + \frac{1}{2}2\zeta\right) = \frac{1}{2}I(2\xi) + \frac{1}{2}I(2\zeta)$$

and use independence.

Clearly if we want state-dependent preferences, we cannot expect monotonicity to hold in this environment.

5.5.3 Gilboa-Schmeidler 1989

Here we replace the second axiom above with:

Axiom 5.5.9 (c-independence)**.** $\forall f \in \mathcal{F}$, $\forall \alpha \in (0,1)$, $\forall \pi \in \Delta(Z)$ *(note that this is a constant), $f \succ g$ iff*

$$\alpha f + (1-\alpha)\pi \succ \alpha g + (1-\alpha)\pi$$

where addition is performed by state.

In the proof above, the only step that changes is the last step, where additivity needs to be replaced by super-additivity. Subsequently, the relevant theorem from convex analysis can be invoked.

5.5.4 Schmeidler 1989

Here we replace the second axiom above with:

Axiom 5.5.10 (Comonotonic Independence)**.** *For all $f, g, h \in \mathcal{F}$, pairwise comonotonic, and for all $\alpha \in (0,1)$,*

$$\alpha f + (1-\alpha)h \succ \alpha g + (1-\alpha)h$$

where addition is performed by state.

Theorem 5.5.11. *Axioms 5.5.1, 5.5.3, 5.5.4, 5.5.6, 5.5.9 above hold iff $\exists u : \Delta(Z) \mapsto \mathbb{R}$ affine and non-constant, $\exists! \nu$ a capacity s.t.*

$$f \mapsto \int u \circ f d\nu \tag{5.7}$$

represents $\succ$. In addition, Axiom 5 holds iff ν is convex.

5.6 Variational Preferences

5.6.1 Maccheroni, Marinacci and Rustichini 2006

Recall that MEU preferences are of the form

$$I(\xi) = \min_{p \in C} \mathbb{E}_p \xi.$$

This is modified here to

$$I(\xi) = \min_{p \in \Delta(S)} \left(\mathbb{E}_p \xi + c(p) \right). \tag{5.8}$$

where $c : \Delta(S) \mapsto [0, \infty]$ is:

1. Convex

2. Grounded: there exists p s.t. $c(p) = 0$

3. Lower semi-continuous

Note that MEU can be expressed as a special case of this framework by defining c as:

$$c(p) = \begin{cases} 0 & p \in C \\ \infty & p \notin C \end{cases}$$

5.6.2 Hansen-Sargent Multiplier Preferences

Preferences will be related to those above with the following cost function. Fix $q \in \Delta(S)$, which will play the role of a best guess, and define the cost function

$$c(p) = R(p||q) = \sum_s \log\left(\frac{p(s)}{q(s)}\right) p(s). \tag{5.9}$$

Axiom 5.6.1 (Unboundedness)**.** *$\exists \pi \succ \rho \in \Delta(Z)$ s.t. $\forall \alpha \in (0,1)$, $\exists \mu \in \Delta(Z)$ s.t. either*

$$\rho \succ \alpha\mu + (1-\alpha)\pi$$

or

$$\alpha\mu + (1-\alpha)\rho \succ \pi.$$

Theorem 5.6.2. *I is:*

1. *concave*
2. *normalized*
3. *monotone*
4. *translation invariant*

(but not positive homogenous) if and only if $\exists c$ convex, grounded and lower semi-continuous s.t.

$$I(\xi) = \min_{p \in C} \left(\mathbb{E}_p \xi + c(p)\right). \tag{5.10}$$

In addition Axiom 5.6.1 holds if and oly if c is unique.

Translation invariance is playing a role similar to that of CARA preferences here.

A weaker version of c-independence is

Axiom 5.6.3 (Weak c-independence)**.** *$\forall f, g, \in \mathcal{F}$, $\forall \pi, \pi' \in \Delta(Z)$, $\forall \alpha \in (0,1)$:*

$$\alpha f + (1-\alpha)\pi \succ \alpha g + (1-\alpha)\pi \implies \alpha f + (1-\alpha)\pi' \succ \alpha g + (1-\alpha)\pi'.$$

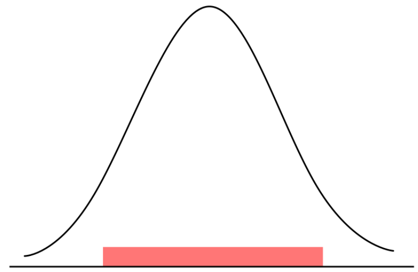


Figure 5.6: The shaded set is represented in a more fuzzy manner by the multiplicative cost function

Note that c-independence can be written as

$$\alpha f + (1-\alpha)\pi \succ \alpha g + (1-\alpha)\pi \iff \beta f + (1-\alpha)\pi' \succ \alpha g + (1-\beta)\pi',$$

and so this definition of a weak version is requiring the statement to be true only when $\alpha = \beta$.

5.6.3 Chateauneuf, Faro 2009

Here we replace translation invariance in the theorem above with positive homogeneity and obtain a representation of the form

$$I(\xi) = \min_p \frac{1}{\psi(p)} \mathbb{E}_p \xi, \tag{5.11}$$

where $\psi : \Delta(S) \mapsto [0,1]$ is a fuzzy set, as represented in Figure 5.6.

5.6.4 CMMM 2011

Theorem 5.6.4. *I is:*

1. *quasi-concave*
2. *normalized*
3. *monotone*
4. *continuous*

(but not positive homogenous or translation invariant) if and only if

$$I(\xi) = \inf_{p \in \Delta(S)} G\left(\mathbb{E}_p, p\right). \tag{5.12}$$

5.6.5 Relative ambiguity aversion

5.6.5.1 Ghirardato-Marinacci 2002

Definition $\succeq_2$ is *more ambiguity averse* than $\succeq_1$ if and only if

$$\pi \succeq_1 f \implies \pi \succeq_2 f$$

$\forall \pi \in \Delta(Z), f \in \mathcal{F}$.

Proposition 5.6.5. *Let $\succeq_1, \succeq_2$ be variational preferences. $\succeq_1$ is more ambiguity averse than $\succeq_2$ iff $u_1 = au_2 + b$ and $c_1 \leq c_2$ (for example in the MEU version, $C_1 \supseteq C_2$).*

5.6.5.2 A stronger version

Note that rank dependent utility (RDU) is a special case of choquet expected utility (CEU) which (if the capacity is convex) is a special case of maxmin expected utility (MEU). By the following definition RDU is strongly ambiguity averse.

Definition $\succeq_1$ is *strongly more ambiguity averse* than $\succeq_2$ iff $\exists x, f$ s.t. $x \succeq_2 f \wedge x \succ_1 f$ and $\succeq_1$ is more ambiguity averse than $\succeq_2$.

5.7 Different sources of uncertainty

There are two primary approaches to evaluating preferences regarding different sources of uncertainty:

1. The first is Anscombe-Aumann, where S and $\Delta(Z)$ are two different sources. Ergan and Gul use $S = S_a \times S_b$ which is a more general version of this. Another extension of this is a paper by Neilson which uses subjective expected utility on S and expected utility on $\Delta(Z)$.

2. A different approach is to consider a sample space with multiple sigma algebras on it: S is associated with $\Sigma_1, \Sigma_2, \Sigma_3, \ldots$. This approach is used in multiple papers by Chew and Sagi. This can capture both the three and two color paradoxes; for the three color version, consider:

$$S = \{R, G, B\}, \Sigma = \{\{R\}, \{G, B\}, S, \emptyset\}$$

For the two color version consider

$$S = \{BB, BW, WB, WW\}$$
$$\Sigma_1 = \{\{BB, BW\}, \{WB, WW\}, S, \emptyset\}$$
$$\Sigma_2 = \{\{BB, WB\}, \{WB, WW\}, S, \emptyset\}$$

5.8 SOEU

5.8.1 The basic set-up

Consider preferences on acts $f : S \mapsto \Delta(Z)$, represented using $u : \Delta(Z) \mapsto \mathbb{R}$ affine, $p \in \Delta(S)$ and $\psi : u(\Delta(Z)) \mapsto \mathbb{R}$ by:

$$V(f) = \int_s \psi(u(f(s)))dp(s). \tag{5.13}$$

Consider in the following an example a difference between known and unknown probabilities in this representation.

Example 5.8.1. *Suppose that $S = \{B, W\}$ and $p(B) = p(W) = \frac{1}{2}$. Consider the lottery $\pi = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$ betting on say a black ball. This is represented by*

$$\psi\left(\frac{1}{2}u(0) + \frac{1}{2}u(1)\right),$$

whereas with unknown probabilities we get the mixture after the concave function is imposed. Consider the act $f = \delta_{0\{B\}}\delta_1$, for example, which is represented as

$$\frac{1}{2}\psi(u(0)) + \frac{1}{2}\psi(u(1)),$$

which captures the difference between uncertainty and ambiguity in this model.

The axioms for this paper are the Anscombe-Aumann axioms without independence, and the Savage axioms on Anscombe-Aumann acts. The imposition of Savage's axioms provide that

$$f \mapsto \int v(f(s))dp(s),$$

with $v : \Delta(Z) \mapsto \mathbb{R}$. VNM on lotteries provides that $\pi \mapsto u(\pi)$ and thus we obtain that $V = \psi \circ u$.

5.8.2 Relationship with Hansen-Sargent

Recall that the Hansen-Sargent representation has the form

$$V(f) = \min_p \int u(f(s))dp(s) + \theta R(p||q),$$

where the cost function is defined in Equation 5.9. It turns out that this is SOEU with

$$\psi(a) = -\exp\left(-\frac{a}{\theta}\right).$$

Moreover, $\theta R(p||q)$ is the only cost function that provides SOEU.

5.8.3 Klibanoff, Marinacci, Mukerji 2005

Here the representation uses $u : Z \mapsto \mathbb{R}$, $\psi : co(u(Z)) \mapsto \mathbb{R}$ and $\mu \in \Delta(\Delta(S))$ with a functional form of

$$V(f) = \int_{\Delta(S)} \psi\left(\int_S u \circ f dp\right) d\mu(p).$$

Note that if

$$\mu = \sum g_i \delta_{\delta_{s_i}},$$

this is the Neilson setting. With this model it is difficult to determine what the content of the model is; in particular it is difficult to interpret what betting on a distribution of an outcome that only occurs once means.

The objects of interest are

$$\mathcal{F} = \{f : S \mapsto \Delta(Z)\}$$

with preferences $\succeq$ on $\Delta(\mathcal{F}) \ni F$. Then using p, a probability on S we full construct a compound lottery using F:

$$\psi(F,p) \in \Delta(\Delta(Z)) \text{ s.t. } \psi(F,p) := F\left(\{f \in \mathcal{F} | \psi(f,p) \in B\}\right)$$

Theorem 5.8.2. *The axioms are that preferences are a weak order, continuity, first and second stage independence, and dominance, i.e.* $\forall F, G \in \Delta(\mathcal{F})$, *if* $\psi(F,p) \succeq \psi(G,p) \forall p \in \Delta(S)$ *then* $F \succeq G$.

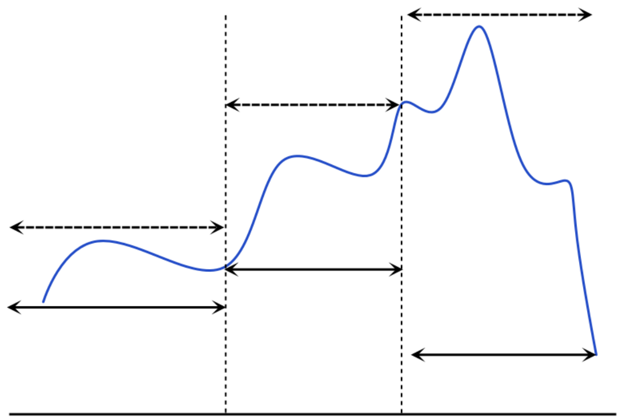


Figure 5.7: Measurable sets are separated by dashed lines, dashed lines with arrows show $\bar{f}$ and solid lines with arrows show $\underline{f}$

With these axioms we obtain that $\succeq$ is represented by

$$F \mapsto \sum_{\mathcal{F}} U(f)F(f) \tag{5.14}$$

$$U(f) = \int_{\Delta(S)} \psi \left(\int_S u(f(s))dp(s) \right) d\mu(p) \tag{5.15}$$

5.9 Expected Uncertain Utility

This is a Gul-Pesendorfer paper, with objects of interest S, Σ with $f : S \mapsto Z$, with two versions of f, $\bar{f}$ and $\underline{f}$ that are Σ-measurable and construct an envelope, as shown in Figure 5.7.

Subsequently there is a utility function $u : \mathbb{R}^2 \mapsto \mathbb{R}$, $p \in \Delta(S, \Sigma)$ with a representation of the form

$$f \mapsto \int u\left(\bar{f}(s), \underline{f}(s)\right) dp(s). \tag{5.16}$$

Lecture 6

Temptation and subjective state spaces

6.1 Initial set up and examples

The references for this section are Blackwell 1957, Koopmans 1964, Kreps 1979, Dekel, Lipman, Rustichini 2001 (DLR), and DLRS 2007 .

We will consider Z finite and preferences $\succ$ on Z. In addition, we will consider $X = 2^Z \setminus \emptyset$ with preferences over *menus* given by $\succ^*$ on X. The classic example for menu choice is the following:

Example 6.1.1. *Suppose one is considering a choice between three restaurants, one which offers both steak and chicken, and two which offer only one option each, and preferences are:*

$$\{st, ch\} \succ^* \{st\} \succ^* \{ch\}$$

which clearly shouldn't happen with ordinary preferences. This is possible, however, if agents gain information from the menu itself.

Definition (Dynamic consistency.): $x \succeq^* x' \iff \exists z \in x \; s.t. \; \forall z' \in x', x \succeq z'$.

Axiom 6.1.2 (Strategic rationality)**.** $x \succeq^* x' \implies x \sim^* x \cup x'$.

Lemma 6.1.3. *Suppose $\succ^*$ is a weak order. Then, $\succeq^*$ satisfies strategic rationality if and only if there exists a weak order $\succ$ on Z s.t. $(\succ^*, \succ)$ satisfy dynamic consistency.*

	ch	st
s_1	0	1
s_2	1	0

Table 6.1: State dependent utility function

Axiom 6.1.4 (Monotonicity, preference flexibility)**.** $x' \subseteq x \implies x' \preceq^* x$.

Consider a representation of the following form: S is a state space, $\pi \in \Delta(S)$, $U : Z \times S \mapsto \mathbb{R}$ where $\succ^*$ is represented by

$$V(x) = \sum_s \pi(s) \max_{z \in x} U(z, s). \tag{6.1}$$

Example 6.1.5. *Consider a binary state space $S = \{s_1, s_2\}$ with probabilities $\pi(s_1) = 0.9$ and $\pi(s_2) = 0.1$ and state dependent utility function as given in Table 6.1. This produces the following overall values to capture the earlier example with menus:*

$$V(\{st\}) = 0.9\,,\; V(\{ch\}) = 0.1\,,\; V(\{ch, st\}) = 1.$$

6.2 Two theorems (Kreps 1979)

Axiom 6.2.1. *$x \sim^* x \cup x' \implies x \cup x'' \sim^* x \cup x' \cup x''$ for all $x'' \in X$.*

Theorem 6.2.2. *Suppose Z is finite. Then, $\succ^*$ on X satisfies:*

1. *Weak order*
2. *Monotonicity*
3. *Axiom 6.2.1*

if and only if there exists S finite and $U : Z \times S \mapsto \mathbb{R}$ such that

$$V(x) = \sum_s \max_{z \in x} U(z, s) \tag{6.2}$$

represents $\succ^*$.

Theorem 6.2.3. *V represents $\succ^*$ if and only if there exists $u : \mathbb{R}^S \mapsto \mathbb{R}$ strictly increasing and*

$$W(x)(s) := \max_{z \in x} U(z, s) \tag{6.3}$$

s.t. $u \circ W$ *represents* $\succ^*$.

Note that the representation in Theorem 6.2.3 is not necessarily additive as is the case with the one in Theorem 6.2.2 as the function u is general.

6.3 DLR, DLRS: Subjective State Spaces

6.3.1 Motivation

One significant problem with the representations considered above is that it is hard to determine what the state space is.

Example 6.3.1. *Consider* $Z = \{z_1, z_2, z_3\}$ *with the rubric that* $x \succ x' \iff \#x > \#x'$. *Consider first the state spaces* $\bar{s} = \{\bar{s}_1, \bar{s}_2, \bar{s}_3\}$ *with* $U(z, s)$ *given by:*

	$\bar{s}_1$	$\bar{s}_2$	$\bar{s}_3$
z_1	2	1	1
z_2	1	2	1
z_3	1	1	2

and note that the same preferences can be represented by a different state space $\hat{s} = \{\hat{s}_1, \hat{s}_2, \hat{s}_3\}$ *with*

	$\hat{s}_1$	$\hat{s}_2$	$\hat{s}_3$
z_1	2	1	0
z_2	0	2	1
z_3	1	0	2

6.3.2 Set up, axioms and definitions

The objects of interest will be $B = \{b_1, \ldots, b_k\}$ and $\succ$ on $2^{\Delta(B)} \setminus \emptyset$. That is, the agent chooses between menus of lotteries instead of menus of deterministic outcomes.

Axiom 6.3.2. $\succ$ *is a weak order.*

Axiom 6.3.3 (Continuity)**.** *For all* $x \in X$, *the sets*

$$\{y|y \succ x\}, \{y|y \prec x\}$$

are open in the Hausdorff topology.

Definition For all x, y and $\lambda \in [0,1]$ the *Minkowsky sum* of two sets is

$$\lambda x + (1-\lambda)y := \{\lambda z + (1-\lambda)z' | z \in x, z' \in y\}$$

That is, we define the convex combination of two sets to be the set of pointwise convex combinations.

Axiom 6.3.4 (Independence). $x \succ y \implies \forall z \forall \lambda \in (0,1)$

$$\lambda x + (1-\lambda)z \succ \lambda y + (1-\lambda)z$$

where addition refers to Minkowsky sums.

Axiom 6.3.5 (Monotonicity). $x \subseteq y \implies x \preceq y$.

Axiom 6.3.6 (L-continuity). *There exist* x^*, x_*, $N > 0$ *s.t. for all* $\epsilon \in \left(0, \frac{1}{N}\right)$ *and* x, y, *s.t.* $d(x,y) \le \epsilon$,

$$((1-N_\epsilon)\, x + N_\epsilon x^* \succeq ((1-N_\epsilon)\, y + N_\epsilon x_*$$

Axiom 6.3.7 (VNM continuity). *For all* $x, y, z \in X$ *where* $x \succ y \succ z$, *there exist* λ, λ' *such that,*

$$\lambda x + (1-\lambda)z \succ y \succ \lambda' x + (1-\lambda')z.$$

Definition An *additive expected utility representation* refers to:

- A measurable space (S, Σ)
- $U : \Delta(B) \times S \longmapsto \mathbb{R}$, $U(\beta, \cdot)$ measurable and $U(\cdot, s)$ of Expected Utility form
- μ on Σ a signed, σ-additive Borel measure
- Preferences $\succ$ are represented by

$$V(x) = \int_S \sup_{\beta \in x} U(\beta, s) d\mu(s). \tag{6.4}$$

6.3.3 Theorems and proof

Theorem 6.3.8. *$\succ$ has an additive EU representation if and only if Axioms 6.3.2, 6.3.4, 6.3.6, 6.3.7 are satisfied. In addition, $\succ$ satisfies Axiom 6.3.5, if and only if μ is positive.*

Theorem 6.3.9. *$\succ$ has an additive EU representation with positive μ if and only if Axioms 6.3.2, 6.3.4, 6.3.5, 6.3.7 are satisfied. Axiom 6.3.6 can be dropped.*

Let X be the set of non-empty, closed, convex subsets of $\Delta(B)$, K the number of prizes in B, and rule out indifference between all objects. Define

$$S^k := \left\{ s \in \mathbb{R}^k | \sum s_i = 0, \sum s_i^2 = 1 \right\}. \tag{6.5}$$

This can be thought of as the set of possible non-trivial specifications of VNM utilities for each element of B.

Definition $\forall x \in X$ define the *support function* $\sigma_x : S^k \to \mathbb{R}$ as

$$\sigma_x(s) = \max_{\beta \in x} \beta s \tag{6.6}$$

Let $C(S^k)$ be the set of continuous real valued functions on S^k, and adopt the convention $f \geq g$ iff $f(S) \geq g(s) \forall s$. Define the set C as

$$C := \left\{ \sigma_x \in C(S^k) | x \in X \right\}$$

Lemma 6.3.10. *C is a convex set and*

$$\sigma_{\left\{\frac{1}{k}, \cdots, \frac{1}{k}\right\}} \equiv 0.$$

We will also define, for all $\sigma \in C$,

$$X_\sigma := \bigcap_{s \in S^k} \left\{ \beta \in \Delta(B) | \beta s \leq \sigma(s) \right\}.$$

Lemma 6.3.11. *The following statements hold:*

1. *For all $x \in X$, $\sigma \in C$*

$$X_{\sigma_X} = X$$

and

$$\sigma_{X_\sigma} = \sigma.$$

2. Mixtures translate:

$$\sigma_{\lambda X+(1-\lambda)Y} = \lambda\sigma_X + (1-\lambda)\sigma_Y.$$

3. The Hausdorff distance translates to the sup-norm on $\mathbb{R}^S$*:*

$$d(X,Y) = ||\sigma_X - \sigma_Y||.$$

4. $x \subseteq y \implies \sigma_X \leq \sigma_Y$.

The proof proceeds in 4 main steps:

1. We begin with three further lemmas:

   Lemma 6.3.12. *Axioms 6.3.2 and 6.3.4 imply that for all* $x \in \Delta(B)$*,* $x \sim cl(x)$*,* $x \sim co(x)$*, that is* x *is indifferent to its closure and convex hull.*

   Lemma 6.3.13. *Axioms 6.3.2, 6.3.4 and 6.3.7 imply that there exists* $V : X \mapsto \mathbb{R}$ *unique up to positive affine transformations.*

   Lemma 6.3.14. V *satisfies L-continuity if and only if* $\succ$ *does.*

2. Here we want to show that

$$V\left(\left\{\frac{1}{k}, \cdots, \frac{1}{k}\right\}\right) = 0$$

   In order to do this define $W : C \mapsto \mathbb{R}$

$$W(\sigma) := V(X_\sigma)$$

   Now W is monotone iff $\sigma \leq \sigma' \implies W(\sigma) \leq W(\sigma')$.

   Lemma 6.3.15. *If* W *is L-continuous and affine, then* $W(0) = 0$.

3. Now we need to extend the domain of this function to invoke the Riesz Representation Theorem.

   Definition Define

$$H := \bigcup_{r \geq 0} rC\,,\; H^* := H - H$$

   where subtraction corresponds to the Minkoswky sense (this means the Minkowsky addition of the sets H and $-H$).

   Lemma 6.3.16. *The following statements are true:*

(a) H^* *is a linear subspace of* $C(S^k)$

(b) $\forall f \in H^* \exists \sigma^1, \sigma^2 \in C, r > 0$ *s.t.* $f = r(\sigma^1 - \sigma^2)$

(c) H^* *is dense in* $C(S^k)$

Now we need to extend the functional W:

Lemma 6.3.17. *Any L-continuous linear functional* $W : C \mapsto \mathbb{R}$ *has a unique continuous linear extension* $\bar{W}$ *to* $C(S^k)$*. If* W *is monotone, then* $\bar{W}$ *is positive, i.e.* $\bar{W}(f) \geq 0$ *if* $f \geq 0$*.*

4. Invoke the Riesz Representation Theorem[1]: $\exists$ a finite signed Borel measure μ on S^k s.t.

$$\bar{W}(f) = \int_{S^k} f(s) d\mu(s). \tag{6.7}$$

If $\bar{W}$ is positive, then μ is positive. Finally, we have

$$V(x) = W(\sigma_x) = \bar{W}(f) = \int_{S^k} \sigma(s) d\mu(s) = \int_S \sup_{\beta \in x} U(\beta, s) d\mu(s). \tag{6.8}$$

□

[1] Tell Tomasz this theorem is horrible at your own peril.

Lecture 7

Random Choice

The following models present explanations for observations of choice of the nature $\{b\} \succ \{a, b\} \succ \{a\}$. In general such explanations fall into one of a few classes, including:

- Preferences are random across time; the second-period self may have a different preference ordering from the first-period self.
- Contemplation is costly; choice from a larger choice set may be more cognitively difficult.
- Preferences are stable but agents encounter problems of temptation and self-control.

7.1 Luce, 1950

The primitives are a finite prize set Z and mapping $\varphi : 2^Z \to \Delta(Z)$ satisfying,

1. $\varphi(x) \in \Delta(Z)$.
2. The support of $\varphi(x) \subset x$.

The Luce axioms are as follows:

Axiom 7.1.1 (Positivity)**.** *The support of $\varphi(x) = x$*

Axiom 7.1.2 (IIA)**.** *For all x containing elements z, z',*

$$\frac{\varphi(z|\{z,z'\})}{\varphi(z'|\{z,z'\})} = \frac{\varphi(z|x)}{\varphi(z'|x)}.$$

In words: the ratio of the choice probabilities for any two elements is independent of the other items under consideration.

Theorem 7.1.3. *φ satisfies Axioms 7.1.1 (Positivity) and 7.1.2 (IIA) if and only if there exists v such that,*

$$\varphi(z|x) = \frac{v(z)}{\sum_{z' \in x} v(z')}.$$

7.2 Gul and Pesendorfer, 2005

The primitives are a finite prize set Z and preferences $\succeq$ defined on $X = 2^Z \backslash \emptyset$.

Example 7.2.1. *Define prize set $Z = \{a, b, c\}$. Let u represent the first-period self's utility, and v represent the second-period self's utility. Consider the following payoff matrix:*

	u	v
a	0	11
b	10	10
c	5	9

Suppose that the second-period self makes the choice. Then ex-ante valuations of choice sets satisfy: $u(\{a,b\}) = u(\{a\}) = 0$ and $u\{b\}) = 10$. It follows that $\{b\} \succ \{a,b\}$, and hence there is value to commitment.

Definition The value function $V^{RS} : X \to \mathbb{R}$ is Random Strotz if there exist u, v such that

$$V^{RS}(x) := \max_{z \in \arg\max_{z' \in x} v(z')} u(z).$$

That is, a second self optimizes using the utility function v; recognizing this, the first period self chooses the "realistic option" that maximizes his utility function u. We can rewrite this also as,

$$W(x) = \max_{z \in x} u(z) - c^{Strotz}(z, x),$$

where c^{Strotz} is a cost of choice defined by,

$$c^{Strotz}(z, x) = \begin{cases} 0 & \text{if } z \in \arg\max_{z' \in x} v(z') \\ \infty & \text{otherwise} \end{cases}$$

Later, we will compare this with the corresponding Gul-Pesendorfer (2001) cost,

$$c^{GP}(z,x) = \max_{z' \in x} v(z') - v(z)$$

The sufficient and necessary axioms for preferences to be represented by the Strotz model are:

Axiom 7.2.2 (Weak Order)**.** *$\succeq$ is a complete and transitive binary relation.*

Axiom 7.2.3 (No Compromise)**.** *$x \sim x \cup y$ or $y \sim x \cup y$.*

(NC) allows for the possibility that $x \succ x \cup y$; that is, the agent strictly prefers a smaller set of alternatives. This is at odds with standard models of decision making, which imply that if $x \succeq y$, then $x \sim x \cup y$. Note that if indeed $x \succ x \cup y$, it must be that the second period choice is from y; therefore $y \sim x \cup y$.

Theorem 7.2.4. *If Z is finite, $\succ$ on X satisfies Axioms 7.2.2 (Weak Order) and 7.2.3 (NC) if and only if there exist $u, v : Z \to \mathbb{R}$ such that $\succ$ is a Strotz preference.*

There are problems with infinite choice sets Z; second-period selves may disagree with each other, resulting in no optimal choice for the agent at period 1.

7.3 Gul and Pesendorfer, 2001

The primitives are: a compact set of prizes B, a set of lotteries over B denoted $\Delta(B)$, a set of menus over lotteries X, and a preference relation $\succ$ over X. Gul and Pesendorfer distinguish between the notion of commitment and the notion of self-control in the following way:

Definition The preference $\succeq$ has a *preference for commitment at the menu* $A \in X$ if there exists $B \subset A$ such that $B \succ A$. The preference $\succeq$ has a *preference for commitment* if $\succeq$ has a preference for commitment at some $A \in X$.

Definition The preference $\succeq$ has *self control at* C if there exist menus $A, B \in X$ such that $C = A \cup B$ and $A \succ A \cup B \succ B$. The preference $\succeq$ has *self-control* if $\succeq$ has self control at some $C \in X$.

7.3.1 Representation with Self-Control

The relevant axioms are:

(A1) *Weak Order.* $\succeq$ is a complete and transitive binary relation.

(A2) *Strong Continuity.* The sets $\{x : x \succeq y\}$ and $\{x : y \succeq x\}$ are closed.

(A3) *Independence.* $x \succ y$ and $\alpha \in (0,1)$ imply $\alpha x + (1-\alpha)z \succ \alpha y + (1-\alpha)z$.

A standard decision maker who experiences no temptation and has no preference for commitment satisfies (A1) - (A3). The following axiom allows for preference for commitment.

(A4) *Set Betweenness.* $x \succeq y$ implies $x \succeq x \cup y \succeq y$.

This axiom captures the notion that an option that is not chosen in period 2 may decrease the utility of the decision-maker by presenting a temptation.

Theorem 7.3.1. *$\succ$ on X satisfies axioms (A1) - (A4) if and only if there exist continuous and linear utility functions u, v such that $\succ$ can be represented by $W^{GP}_{u,v}$:*

$$W^{GP}_{u,v}(x) = \max_{\beta \in x} \underbrace{u(\beta)}_{\textit{commitment utility}} - \underbrace{[max_{\alpha \in x} v(\alpha) - v(\beta)]}_{\textit{cost of self control}}$$

The two utility functions u and v represent the agent's first-period and second-period utilities. Gul and Pesendorfer refer to the first as a *commitment ranking* and the latter as a *temptation ranking.* Note that the utility $W^{GP}_{u,v}(x)$ represents utility from the first period perspective ($\succ$ is a period one preference).

Note also that the agent chooses from the menu the item which maximizes $u + v$. In doing so, he shows partial self control by compromising between u and v at the choice stage instead of simply maximizing v. The ability to do this is weakened in the following section, in which we present machinery to describe agents with no self control.

7.3.2 Overwhelming Temptation

Following, Gul and Pesendorfer replace (A2) with a weaker notion of continuity, consisting of three components:

(A2a) *Upper Semi-Continuity.* The sets $\{B \in X : B \succeq A\}$ are closed.

(A2b) *Lower von Neumann-Morgenstern Continuity.* $A \succ B \succ C$ implies $\alpha A + (1-\alpha)C \succ B$ for some $\alpha \in (0,1)$.

(A2c) *Lower Singleton Continuity.* The sets $\{x : \{y\} \succeq \{x\}\}$ are all closed.

Theorem 7.3.2. *The binary relation $\succeq$ satisfies (A1), (A2a)-(A2c), (A3), and (A4) if and only if there are continuous linear functions u, v such that either the function W defined as*

$$W^{GP}(x) := \max_{\beta \in x}\{u(\beta) + v(\beta)\} - \max_{\alpha \in x} v(\alpha)$$

for all $x \in X$ or the function W defined as,

$$W^{GP}(x) := \max_{\beta \in x} u(\beta) \quad \textit{subject to} \quad v(\beta) \geq v(\alpha), \quad \textit{for all } \alpha \in x$$

for all $x \in X$ represents $\succeq$.

Corollary 7.3.3. *The temptation preference $\succeq$ with representation (u, v) has no self-control if and only if the function W defined as*

$$W^{GP}(x) := \max_{\beta \in x} u(\beta) \quad \textit{subject to} \quad v(\beta) \geq v(\alpha), \quad \textit{for all } \alpha \in x$$

for all $x \in X$ represents $\succeq$.

7.4 Dekel and Lipman, 2011

Dekel and Lipman (2011) demonstrate an equivalence between the random Strotz representation and the Gul and Pesendorfer representation (every preference with a GP-representation also has a random Strotz representation). It follows that one cannot use preferences over menus to distinguish self control from random temptation.

First, generalize random Strotz preferences as follows:

$$V^{RS} = \int_{\mathbb{R}^z} \max_{\beta \in B_v(x)} u(\beta) d\mu(v).$$

Theorem 7.4.1. *For all $W^{GP}_{u,v}$, there exists $\mu \in \Delta(\mathbb{R}^z)$ such that $W^{GP}_{u,v} = V^{RS}_{u,v}$.*

7.5 Stovall, 2010

Example 7.5.1. *Consider behavior of the nature $b \succ bp \sim bc \succ bpc$. What could be the rationale?*

1. *Temptation is higher with 2 alternatives.*

2. *Agents have random cravings.*

This paper models the latter. That is, agents have multiple cravings, to each of which they respond like a Gul-Pesendorfer agent. The primitives are a finite set of outcomes Z, a set of probability distributions over outcomes Δ, lotteries $\beta \in \Delta$, a set X of closed nonempty subsets of $\Delta(Z)$, menus $x \in X$, and preference relation $\succeq$ over X.

Axiom 7.5.2 (Weak Order)**.** *$\succeq$ is complete and transitive.*

Axiom 7.5.3 (Continuity)**.** *The sets $\{x|x \succeq y\}$ and $\{x|y \succeq x\}$ are closed in the Hausdorff topology.*

Axiom 7.5.4 (Independence)**.** *If $x \succ y$, then for every $z \in X$ and $\lambda \in (0, 1]$,*

$$\lambda x + (1 - \lambda)z \succ \lambda y + (1 - \lambda)z,$$

where addition refers to Minkowsky sums defined previously.

Axiom 7.5.5 (Finiteness)**.** *Every menu has a finite critical subset.*

Axiom 7.5.6 (Exclusion)**.** *If $\{\beta\} \succeq \{\alpha\}$ for every $\beta \in x$, then $x \succeq x \cup \{\alpha\}$.*

This axiom regards adding a *bad* alternative to a *good* menu.

Axiom 7.5.7 (Inclusion)**.** *If $\{\alpha\} \succeq \{\beta\}$ for every $\beta \in x$, then $x \cup \{\alpha\} \succeq x$.*

This axiom regards adding a *good* alternative to a *bad* menu.

Definition A *temptation representation* is a function $W(x)$ such that,

$$W(x) = \sum_{i=1}^{n} [\max_{\beta \in x} u(\beta) + v_i(\beta) - \max_{\alpha \in x} v_i(\alpha)] p_i$$

where $p_i > 0$ for all i, $\sum_{i=1}^{n} p_i = 1$, and u, v_i are von Neumann-Morgenstern Expected Utility functions.

Note that i is a subjective state, v_i is the relevant temptation to which the agent assigns probability p_i, and $u+v_i$ represents choice preference in state i. When there is only one state, this reduces to the Gul-Pesendorfer representation.

Theorem 7.5.8. *$\succeq$ satisfies Axioms 1-6 if and only if $\succeq$ has a temptation representation.*

7.6 Noor and Takeoka, 2010

This paper extends Gul and Pesendorfer (2001) by allowing for convex self-control costs.

Definition A *convex self-control representation* is a function $W(x)$ such that,

$$W(x) = \max_{\beta \in x} u(\beta) - \phi(\max_{\alpha \in x} v(\alpha) - v(\beta))$$

for some increasing convex function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$.

7.7 Gul and Pesendorfer, 2004

This paper extends Gul and Pesendorfer (2001) to an infinite horizon. The main result is a representation theorem yielding a utility function of the following form:

$$W(z) = \max_{\beta \in Z} \int (u(c) + v(c) + \delta W(z'))d\beta(c, z') - \max_{\alpha \in z} \int v(c)d\alpha(c, z').$$

As before, there is a cost for exercising self-control. Optimal behavior trades-off the long-run interest of the individual, represented by $u + \delta W$, with the temptation to consume.